الجمهورية الجزائرية الديمقراطية الشعبية
وزارة التعليم العالي و البحث العلمي

République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université Ahmed Zabana de Relizane
Faculté des Sciences et de la Technologie
Département de Mathématiques

جامعة أحمد زبانة - غليزان
Ahmed Zabana University-Relizane

MEMOIRE

En vue de l'obtention du diplôme de MASTER en :

Géométrie différentielle

Intitulé

Introduction to Fractional Calculus and Its Applications in Applied Mathematics and Other Sciences

Présenté par :

Mr :BENAMAR AISSA el arabi

Mr : Tama Abdelkader

Devant les membres de jury :

Président : Mr Medjahed Djilali Maître de conférence A (U. Relizane)

Encadreur : Mr Kherraz Tahar Maître de conférence B (U. Tissemsilt)

Examinateur : Mr Moustafa Beddani Maître de conférence A (U. Mostaganem)

Année universitaire : 2024/2025

إهـــداء

إلى من كانوا الضوء في عتمي، والسند في تعبي، والدافع في لحظات الشك... إلى الذين لولاهم لما كنت أصل إلى هذا المقام العالي الذي طالما حلمت به. إلى أبي، الذي كنت أجد فيه السند الذي أسندت عليه أحلامي، معلماً بالصبر؛ ورباني على الكرامة والعطاء. دعاؤه الصامت كان نبراساً من طمأنينة لا ينضب.

وإلى أمي، التي وهبتني من قلبها نوراً، ومن روحها دفعاً، يا من كنتِ الدعاء الذي لا يغيب، والسند الذي لا يتبدل، في كل خطوة انتصار، أذكركِ أول ما يخطر في البال.

إلى زوجي الحبيبة، رفيقة الدرب، وشريكة الحمل والنصب من صبرت على غيابي، واحتملت مزاجي وتقلباتي، من كانت لي سكناً وطمأنينة، وداعماً بصمت لا يُقارن. إلى فلدة كبدي و ابني الغالي أنت النور القادم، والدافع الأعظم، رغم صغرك كنت حافزي لأصنع لك مستقبلاً تفخر به، أهديك هذه الثمرة لتكون أول إبن في إرث علمي اتركه لك

إلى إخوتي وأخواتي، كنتم خير سند، وخير رفيق، لكني أخجل من تقصيري وامتناني.

إلى أساتذتي الكرام، لكم أرفع قلبي امتناناً، فقدتموني منحة العلم، وغرستم في روحي حب البحث والسعي وراء المعرفة.

إلى زملائي وأصدقائي، أذكر محطات هذه الرحلة، لن تُنسى أيام الدراسة التي جمعتنا، ولن تُنسى المواقف الصادقة.

وإلى كل من كان له أيّ دور، ولو بكلمة، ولو بدعاء، أهديكم هذه الرسالة، عربون وفاء وامتنان، فنجاحي هذا، هو ثمرة حبكم ودعمكم.

بن عمار عيسى العربي

إهـــداء

إلى من كانوا نبراس طريقي ومصدر قوتي في هذه الرحلة العلمية...

إلى والديَّ الكريمين، اللذين زرعا في قلبي حب العلم ، وربّياني على قيم الصبر والاجتهاد، وكانا الدعامة الأولى في كل خطوة أخطوها، فلهما مني كل التقدير والامتنان والدعاء الصادق.

إلى إخوتي وأخواتي، الذين كانوا لي عوناً وسنداً في السراء والضراء، إليكم محبتي الخالصة وامتناني العميق.

إلى أبنائي، فلذات كبدي، الذين أستمدّ من ابتسامتهم القوة وأتحمّل مشقة الأيام، أرجو أن يكون هذا الإنجاز بذرة تفتحون بها ذات يوم.

إلى كل من ساندني، ووقف جنبي بكلمة، أو دعاء، أو ابتسامة،

أهديكم هذه الثمرة، عربون وفاء وتقدير.

مع فائق التحية والتقدير،

تامّة عبد القادر

Acknowledgements

First of all, I would like to thank the Almighty Allah for giving me the patience, the will and the energy to complete this work.

I would like to express my deepest gratitude to my supervisor, Dr. **Tahar KHERRAZ**, for the trust he placed in me by accepting to oversee this thesis. His invaluable guidance, unwavering support, and the countless hours he devoted to directing this research work were instrumental in its completion.

I am also profoundly grateful for his constant availability, punctuality in meeting deadlines, and meticulous proofreading of the documents I submitted. Above all, I deeply admire his exceptional human qualities—his patience, attentiveness, and understanding throughout this research journey.

This work would not have been possible without his expertise and encouragement, and for that, I am sincerely thankful.

I would also like to thank Professor **Medjahed Djilal** for doing me the honor of chairing the jury for this thesis.

My sincere thanks also go to Dr. **Mostapha Badani** for the honor they bestowed on me by agreeing to serve on the jury. Their participation is greatly appreciated.

I would like to thank all the members of the Department of Mathematics of the Faculty of Sciences and Technology of the Ahmed Zabana University of Relizane:

Finally, I would like to express my gratitude to all those who have contributed to this thesis, whether scientifically or through their encouragement or affection. I would also like to thank all those who helped me in any way.

Benamar aissa Elarabi
Tama Abdelkader

Contents

Introduction

The history of fractional differential calculus has a long time. The birth of idea of fractional calculus can be seen as a generalization of the ordinary differentiation and integration to fractional numbrer order, this is the important idea of fractional calculus. The question of derivatives of non-integer order was first mentioned in 1695 by Leibniz in a letter to L'Hospital. However, when l'Hospital asked him what the derivative of order one-half of the function x might be, Leibniz replied that it led to a paradox from which useful consequences would one day be drawn. More than 300 years later, we are only just beginning to overcome these difficulties. Many mathematicians have studied this question, including Euler (1730), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), and others. Different approaches have been used to generalize the concept of differentiation to non-integer orders:

- The limit of the difference quotient of a function is generalized in the form of the Grünwald-Letnikov formula, which is very useful numerically. - Integration, as the inverse operation, via Liouville's integral formula, leads to the Riemann-Liouville and Caputo formulas. - Finally, the Fourier and Laplace transforms associate fractional differentiation with multiplication by $(2i\pi\nu)^{\alpha}$ or p^{α}, where α is non-integer.

However, for a long time, these different definitions seemed not to always yield the same results. This apparent inconsistency could only be resolved thanks to the framework.

In recent years, there has been a significant development in the theory of fractional differential equations. It is caused by its applications in the modeling of many phenomena in various fields of science and engineering such as acoustic, control theory, chaos and fractals, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, optics, economics, astrophysics, chaotic dynamics, statistical physics, thermodynamics, proteins, biosciences, bioengineering, etc. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes, see for example [10, 20, 22].

In the context of this work, we have organized this thesis in tree chapters as follows:

In **Chapter 1**, we introduce notations and some basic fundamental concepts of nonlinear analysis, fractional calculus such as Riemann-Liouville fractional integrals, Caputo fractional derivative, Hadamard-Caputo fractional calculus. We end this chapter by providing numerical examples to support the obtained fractional calculus results.

In **Chapter 2**, we are concerned to notations and some basic fundamental concepts of nonlinear analysis,We also recall a number of fixed-point theorems used to establish the existence results for the proposed problems. Included among the fixed-point theorems recognized by their names are Banach's contraction principal, Sadovskii's fixed point theorem, Monch's fixed-point theorem a.

In **Chapter 3**, we study the existence and uniqueness of solutions for fractional boundary value problems (FBVP) involving Caputo-Hadamard fractional derivatives and Anti-Periodic Conditions we will give existence and uniqueness results for the followings problems of fractional differential equations:

$$ {}^{C}_{H}D^{\alpha}x(t) = f(t, x(t)), \quad 0 < \alpha < 1,\ 1 < t < T, \tag{1} $$

$$ ax(1) + bx(T) = \lambda I^{\sigma}x(s)ds, \quad 0 < \sigma < 1,\ \lambda \in \mathbb{R}, \tag{2} $$

where ${}^{C}_{H}D^{\alpha}$ denote the Caputo-Hadamard fractional derivative of order $\alpha\}$,
$0 < \alpha < 1$ and $f : I \times E \to E$ is a given continuous functions. Here, E is a Banach space with norm $\|\cdot\|$ and $I = [1, T]$, a,b ,λ are real constants and $1 < s < T$.

More exactly, we prove the existence of solutions for the above problem using Schauder's fixed point theorem nonlinear alternative for single valued maps, and Scheafer's fixed point theorem are given. Also, we preve the uniqueness results by means of Boyd and Wong's and Banach's fixed point theorems. The obtained result is illustrated by an example.

Basic Properties of Fractional Derivatives 1

Abstract. This chapter contains a brief visit to the origin of Fractional Calculus, notations, definitions of functional spaces, fractional integrals and fractional derivatives.There for explores the diverse fractional derivatives, presenting their definitions, key characteristics, and domains of applicability. From the well-known Caputo and Riemann-Liouville formulations to more specialized operators like the Hadamard, derivatives, each variant offers distinct advantages for modeling complex phenomena such as: Memory-dependent systems,Non-local dynamics,Anomalous diffusion Fractal processes Then we give Somme applications of Fractional calculus.

1.1 The Birth and Development of Fractional Calculus

The French mathematician L'Hôpital wrote to Leibniz in 1695 regarding the extension of derivatives: Can the concept of integer-order derivatives be generalized to remain valid for non-integer orders?

Following this unprecedented discussion, the subject of fractional calculus caught the attention of other great mathematicians, many of whom directly or indirectly contributed to its development.

Following this groundbreaking discussion, the subject of fractional calculus captured the attention of many great mathematicians, each contributing directly or indirectly to its evolution.

September 30, 1695, marks the birth of fractional calculus—a field originating from the correspondence between Leibniz and L'Hôpital. Over the centuries, it was advanced by the works of: *Euler* (1730), and *Lagrange* (1772). Over the years, *Laplace* (1812), *Fourier* (1822), *Abel* (1823), *Liouville* (1832), *Riemann* (1847), Grünwald (1867), *Letnikov* (1868), *Hadamard* (1892), *Riesz* (1922), *Kober* (1940), *Zygmund* (1945), *Kuttner* (1953), and *Liverman* (1964)...

Leonhard Euler in 1783 made his first significant remarks on fractional-order derivatives. His work on numerical progressions led to the groundbreaking generalization of factorials through the **Gamma function**, providing essential mathematical tools for fractional calculus.

Building on these foundations, Joseph-Louis Lagrange in 1772 - just over fifty years after Leibniz's death - made indirect but crucial contributions. He extended the law of exponents for integer-order differential operators, establishing principles that would later be adapted to fractional orders under specific conditions.

The field took a major step forward in 1812 when Pierre-Simon Laplace formulated the first rigorous definition of fractional derivatives. Laplace demonstrated that such derivatives could be defined for functions representable by integrals, which in modern notation we would express as:

$$\int y(t)t^{-x}dt$$

Shortly thereafter, Sylvestre François Lacroix advanced the theory by generalizing the derivative of power functions. He extended the integer-order derivative of $y(t) = t^m$ (where $m \in \mathbb{N}$) to fractional orders, creating an important bridge between classical and fractional calculus.

1.2 Useful mathematical functions

Before looking at the definition of Fractional Derivative or Integral, we will first discuss some useful mathematical definitions that are inherently tied to fractional calculus and will commonly be encountered. These include the Gamma function, the Beta function, the Error function, the Mittag-Leffler function, and the Mellin-Ross function.

1.2.1 Gamma Function

The fractional calculation is based on Euler's gamma function. It generalizes the factorial $n!$ and allows n to take real or even complex values. Let $z \in \mathbb{R}$, we have:

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt.$$

The function $g : t \to t^z e^{-t}$ is continuous and positive over $]0, +\infty[$. For any natural number $n \geq 2$, we have:

$$\Gamma(n) = (n-1)\Gamma(n-1).$$

Also,

$$\Gamma(1) = \int_0^{+\infty} e^{-t}dt = [-e^{-t}]_0^{+\infty} = 1.$$

Thus, we obtain:

$$\forall n \in \mathbb{N}^*, \Gamma(n) = (n-1)!.$$

For $\Gamma\left(\frac{1}{2}\right)$, let $u = \sqrt{t}$ and $dt = 2udu$, which implies:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}}dt = \int_0^{+\infty} \frac{e^{-u^2}}{u}2udu = 2\int_0^{+\infty} e^{-u^2}du = \sqrt{\pi}.$$

1.2.2 Beta Function

The fractional calculation is also based on the Beta function, which plays an important role in certain combinations with the Gamma function.

Definition 1.1 *(see [10])*
The Beta function is defined by:

$$B(z,w) = \int_0^{+\infty} t^{z-1}(1-t)^{w-1}dt, \quad \Re(z) > 0, \quad \Re(w) > 0.$$

The relation between the Gamma function and the Beta function is:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

1.2.3 Mittag-Leffler Function

Integer-order differential equations are also based on a function called the Mittag-Leffler function, which plays a very important role in theory. It facilitates the study of the existence of solutions for fractional differential equations.

Definition 1.2 *(see [?]). For $x \in \mathbb{C}$ such that $\Re(x) > 0$, the Mittag-Leffler function is defined as:*

$$M_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)},$$

If $\alpha = 1$, we obtain the exponential function:

$$M_1(x) = e^x.$$

The Mittag-Leffler function can be generalized for two parameters:

$$M_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0.$$

1.3 Fractional Integral of Riemann-Liouville

Definition 1.3 *Let $\Omega = [a,b] \subset \mathbb{R}$ be finite and $g \in L^p(\Omega)$. The fractional integrals of Riemann-Liouville, $I^\alpha_{a+}g$ and $I^\alpha_{b-}g$, of real order $\alpha > 0$ are defined as:*

$$I^\alpha_{a+}g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds, \quad (t > a, \alpha > 0).$$

$$I^\alpha_{b-}g(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} g(s)ds, \quad (t < b, \alpha > 0).$$

Where $\Gamma(\alpha)$ is the Gamma function. The formula I^α_{a+} is called the left-sided fractional integral of order α, and I^α_{b-} is called the right-sided fractional integral of order α.

If $\alpha = n \in \mathbb{N}$, the fractional integral of Riemann-Liouville takes the form:

$$I_{a+}^{n} g(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} g(s) ds, \quad (n \in \mathbb{N}).$$

$$I_{b-}^{n} g(t) = \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} g(s) ds, \quad (n \in \mathbb{N}).$$

1.4 Fractional Derivative in the Sense of Riemann-Liouville

Definition 1.4 *see [20] Riemann-Liouville Fractional Derivative*

The Riemann-Liouville (RL) fractional derivative is a generalization of the standard derivative to non-integer (fractional) orders. It is defined using an integral transform.

For a function $g(t)$ defined on $[a, t]$, the Riemann-Liouville fractional derivative of order α (where $n-1 < \alpha < n$, and $n \in \mathbb{N}$) is given by:

$$D_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{g(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,$$

where: - $\Gamma(\cdot)$ is the Gamma function, - $n = \lceil \alpha \rceil$ (the smallest integer greater than or equal to α), - a is the lower limit of integration (often taken as 0 or $-\infty$).

Example 1.1 *For $f(t) = t^{\beta}$ ($\beta > -1$) and $a = 0$:*

$$D_{0+}^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}.$$

Remark 1.1 *The fractional derivative of Riemann-Liouville, $D_{a+}^{\alpha} g$, of real order $\alpha \geq 0$ is defined as:*

$$D_{a+}^{\alpha} g(t) = \frac{d^n}{dt^n} \left(I_{a+}^{n-\alpha} g(t)\right),$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} g(s) ds, \quad t > a.$$

Where $n = \lceil \alpha \rceil + 1$, and $\lfloor \cdot \rfloor$ is the integer part of a real number. In particular, if $\alpha = n \in \mathbb{N}$, we obtain:

$$D_{a+}^{n} g(t) = \frac{d^n}{dt^n} g(t).$$

$$D_{a+}^{0} g(t) = g(t), \quad D_{a+}^{n} g(t) = g^{(n)}(t),$$

where $g^{(n)}(t)$ denotes the usual derivative of order n of $g(t)$. If $0 < \alpha < 1$, then:

$$D_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} g(s) ds, \quad (t > a).$$

1.4.1 Properties

If g is continuous for $t > a$, then the fractional integration of arbitrary real order defined by (1.1) has the following property:

$$I_{a+}^{\alpha}(I_{a+}^{\beta}g(t)) = I_{a+}^{\alpha+\beta}g(t), \quad (\alpha > 0, \beta > 0),$$

obviously, we can interchange α and β as:

$$I_{a+}^{\alpha}(I_{a+}^{\beta}g(t)) = I_{a+}^{\beta}(I_{a+}^{\alpha}g(t)) = I_{a+}^{\alpha+\beta}g(t), \quad (\alpha > 0, \beta > 0).$$

The most important property of the fractional derivative in the Riemann-Liouville (R-L) sense, for $\alpha > 0$, and $t > a$ is:

$$D_{a+}^{\alpha}I_{a+}^{\alpha}g(t) = g(t),$$

If $\alpha > \beta > 0$, and $g(t) \in L^p(a,b)$, $(1 \leq p \leq \infty)$, then

$$D_{a+}^{\beta}I_{a+}^{\alpha}g(t) = I_{a+}^{\alpha-\beta}g(t),$$

almost everywhere on $[a,b]$, where

$$L^p[a,b] = \left\{ g : [a,b] \to \mathbb{R}; g \text{ is measurable on } [a,b] \text{ and } \int_a^b |g(t)|^p dt < \infty \right\}.$$

In particular, if $\beta = k \in \mathbb{N}$, and $\alpha > k$, then:

$$D_{a+}^{k}I_{a+}^{\alpha}g(t) = I_{a+}^{\alpha-k}g(t),$$

Let $\alpha \geq 0, m \in \mathbb{N}$ and $D = \frac{d}{dt}$. If both fractional derivatives $D_{a+}^{\alpha}g(t)$, $D_{a+}^{m}g(t)$ exist, then we have:

$$D_{a+}^{m}D_{a+}^{\alpha}g(t) = D_{a+}^{\alpha+m}g(t).$$

$$I_{a+}^{\alpha}D_{a+}^{\alpha}g(t) = g(t) - \sum_{j=1}^{n}\left[D_{a+}^{\alpha-j}g(t)\right]_{t=a}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}.$$

Generally, we observe that fractional derivatives and fractional integrals in the **Riemann-Liouville (R-L) sense** of the same order do not commute.

We also have the following composition formulas for $m-1 \leq \alpha < m$ and $n-1 \leq \beta < n$:

$$D_{a+}^{\alpha}D_{a+}^{\beta}g(t) = D_{a+}^{\alpha+\beta}g(t) - \sum_{j=1}^{n}\left[D_{a+}^{\beta-j}g(t)\right]_{t=a}\frac{(t-a)^{\alpha-j}}{\Gamma(1-\alpha-j)}.$$

And,

$$D_{a+}^{\beta}D_{a+}^{\alpha}g(t) = D_{a+}^{\alpha+\beta}g(t) - \sum_{j=1}^{n}\left[D_{a+}^{\alpha-j}g(t)\right]_{t=a}\frac{(t-a)^{\beta-j}}{\Gamma(1-\beta-j)}.$$

From equations (1.13) and (1.14), we conclude that fractional derivatives in the **R-L sense** do not commute.

Example 1.2 *We will calculate the fractional integral $I^{\alpha}_{a+}g(t)$ in the **R-L sense** for the power function $g(t) = (t-a)^{\beta}$, where β is a real number. We use formula (1.1):*

$$I^{\alpha}_{a+}(t-a)^{\beta} = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(s-a)^{\beta} ds.$$

We assume $\beta > -1$ for the convergence of the integral. From (1.15), using the change of variable $s = a + \varepsilon(t-a)$ and employing the Beta function, we deduce:

$$I^{\alpha}_{a+}(t-a)^{\beta} = \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha+\beta} \int_0^1 (1-\varepsilon)^{\alpha-1}\varepsilon^{\beta} d\varepsilon.$$

Where $\varepsilon = 0$ if $s = a$, $\varepsilon = 1$ if $s = t$, and $\varepsilon = \frac{s-a}{t-a}$.
Thus,

$$I^{\alpha}_{a+}(t-a)^{\beta} = \frac{1}{\Gamma(\alpha)} B(\beta+1, \alpha)(t-a)^{\alpha+\beta}.$$

Using the Beta function formula $B(x, w) = \frac{\Gamma(x)\Gamma(w)}{\Gamma(x+w)}$, we obtain:

$$I^{\alpha}_{a+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \quad (\alpha > 0, \beta > -1).$$

*From [28], we compute the fractional derivative $D^{\alpha}_{a+}g(t)$ in the **R-L sense** for the function $g(t) = (t-a)^{\beta}$. Assuming $0 < n-1 \leq \alpha < n$, recall that the definition of the **fractional derivative in the R-L sense** is:*

$$D^{\alpha}_{a+}g(t) = \frac{d^n}{dt^n}(I^{n-\alpha}_{a+}g(t)), \quad (n-1 \leq \alpha < n).$$

We need to suppose $\beta > n$ for the convergence of integral (1.1). Thus:

$$D^{\alpha}_{a+}(t-a)^{\beta} = \frac{d^n}{dt^n}\left(I^{n-\alpha}_{a+}(t-a)^{\beta}\right).$$

Using formulas (1.17) and (1.19), we obtain:

$$D^{\alpha}_{a+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)} \frac{d^n}{dt^n}(t-a)^{\beta+n-\alpha}.$$

We also conclude:

$$\frac{d^n}{dt^n}(x-a)^{\beta+n-\alpha} = (\beta+n-\alpha)(\beta+n-\alpha-1)\cdots(\beta-\alpha+1)(t-a)^{\beta-\alpha}.$$

Thus,

$$\frac{\Gamma(\beta+n-\alpha+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.$$

Using result (1.22) in (1.20), we find:

$$D^{\alpha}_{a+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.$$

*Therefore, the fractional derivative in the sense of **Riemann-Liouville** for the function $g(t) = (t-a)^{\beta}$ is:*

$$D^{\alpha}_{a+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.$$

where $0 \leq n-1 \leq \alpha < n$, $\beta > n$.

Corollary 0.1

Let $\alpha > 0$ and $n = \lfloor \alpha \rfloor + 1$. The equation $(D^{\alpha}_{a+}g)(t) = 0$ holds if:

$$g(t) = \sum_{j=1}^{n} x_j (t-a)^{\alpha-j},$$

where $x_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ are arbitrary constants.
In particular, if $0 < \alpha \leq 1$, the relation $(D^{\alpha}_{a+}g)(t) = 0$ holds if and only if:

$$g(t) = x(t-a)^{\alpha-1}, \quad \forall x \in \mathbb{R}.$$

Example 1.3 *Fractional derivative of a constant. If we take $\beta = 0$ with $\alpha \geq 0$ in equation (1.24), we conclude that the fractional derivative of a constant in the sense of **Riemann-Liouville** is nonzero:*

$$D^{\alpha}_{a+}(x) = \frac{x}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \quad (0 < \alpha < 1).$$

On the other hand, for $j = 1, \ldots, \lfloor \alpha \rfloor + 1$, we have:

$$D^{\alpha}_{a+}(t-a)^{\alpha-j} = 0.$$

Definition 1.5 *(see [?]). For $x \in \mathbb{C}$ such that $\Re(x) > 0$, the Mittag-Leffler function is defined as:*

$$M_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)},$$

If $\alpha = 1$, we obtain the exponential function:

$$M_1(x) = e^x.$$

The Mittag-Leffler function can be generalized for two parameters:

$$M_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \beta > 0.$$

$$I_{a+}^{\alpha} D_{a+}^{\alpha} g(t) = g(t) - \sum_{j=1}^{n} \left[D_{a+}^{\alpha-j} g(t) \right]_{t=a} \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}.$$

Generally, we observe that fractional derivatives and fractional integrals in the **Riemann-Liouville (R-L) sense** of the same order do not commute.

We also have the following composition formulas for $m-1 \leq \alpha < m$ and $n-1 \leq \beta < n$:

$$D_{a+}^{\alpha}(x) = \frac{x}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \quad (0 < \alpha < 1).$$

On the other hand, for $j = 1, \ldots, \lfloor \alpha \rfloor + 1$, we have:

$$D_{a+}^{\alpha}(t-a)^{\alpha-j} = 0.$$

1.5 Fractional Derivative in the Caputo Sense

The fractional derivative of Caputo type and its properties.

Definition 1.6 *(see[10]) The Caputo fractional derivative ${}^{C}D_{a+}^{\alpha} g(t)$ of order $\alpha \geq 0$ on a finite interval $[a, b]$ can be defined using the Riemann–Liouville fractional derivative as:*

$${}^{C}D_{a+}^{\alpha} g(t) = D_{a+}^{\alpha} \left[g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(t-a)^{k} \right],$$

and

$${}^{C}D_{b-}^{\alpha} g(t) = D_{b-}^{\alpha} \left[g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(b)}{k!}(b-t)^{k} \right],$$

where

$$n = \lfloor \alpha \rfloor + 1 \quad \textit{for } \alpha \notin \mathbb{N}, \quad n = \alpha \in \mathbb{N}.$$

The two derivatives are respectively called the left and right Caputo derivatives.

$${}^{C}D_{a+}^{\alpha} g(t) = D_{a+}^{\alpha}[g(t) - g(a)],$$

and

$${}^{C}D_{b-}^{\alpha} g(t) = D_{b-}^{\alpha}[g(t) - g(b)].$$

Theorem 1.1 *(see [32])*

. Let $\alpha \geq 0$, and n be given by (1.27).

If $g \in AC^{m}[a, b]$, the Caputo fractional derivative exists almost everywhere on $[a, b]$,

(a) If $\alpha \notin \mathbb{N}$, the Caputo derivative $D_{a+}^{\alpha} g(t)$ is given by:

$${}^{c}D_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-s)^{n-\alpha-1} g^{(n)}(s) ds$$

$$= (I_{a+}^{n-\alpha} D_{a+}^{n} g)(t).$$

where

$$D = \frac{d}{dt}, \quad n = \lfloor \alpha \rfloor + 1, \quad AC^m[a,b] = \{g : [a,b] \to \mathbb{C} \text{ and } D^{n-1}g \in AC[a,b]\}.$$

If $0 < \alpha < 1$ *and* $g \in AC[a,b]$, *we obtain:*

$$^{C}D_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} g'(s)\, ds = \left(I_{a+}^{1-\alpha} D_{a+}^{1} g \right)(t).$$

(b) *If* $\alpha = n \in \mathbb{N}$, *we obtain:*

$$^{C}D_{a+}^{\alpha} g(t) = g^{(n)}(t).$$

In particular:

$$^{C}D_{a+}^{0} g(t) = g(t).$$

1.5.1 Properties

(See [23], [28]) on fractional derivatives in the sense of Caputo.

If $\alpha \notin \mathbb{N}$ and $g(a) = g'(a) = \cdots = g^{(n-1)}(a) = 0$, with $n = \lfloor \alpha \rfloor + 1$, then the Caputo derivative coincides with the Riemann-Liouville derivative, i.e.:

$$^{C}D_{a+}^{\alpha} g(t) = {}^{R-L}D_{a+}^{\alpha} g(t).$$

If $\alpha \in \mathbb{N}$ and the usual derivative $g^{(n)}(t)$ exists, then the Caputo fractional derivative of order n coincides with $g^{(n)}(t)$:

$$^{C}D_{a+}^{\alpha} g(t) = g^{(n)}(t).$$

The relation between the Riemann-Liouville fractional integral and the Caputo fractional derivative is as follows.

Let $\alpha > 0$ and n be given by (1.27). If $g \in AC^m[a,b]$, then:

$$I_{a+}^{\alpha}\, {}^{C}D_{a+}^{\alpha} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!} (t-a)^k.$$

If $0 < \alpha < 1$ and $g \in AC[a,b]$, we find:

$$I_{a+}^{\alpha}\, {}^{C}D_{a+}^{\alpha} g(t) = g(t) - g(a).$$

Example 1.4 *Let the function* $g(t) = (t-a)^{\beta}$. *We calculate* ${}^{C}D^{\alpha}_{a+}g(t)$ *in the sense of Caputo.*

To proceed, we assume that $0 \leq n-1 \leq \alpha \leq n$, *and recall that the definition of the Caputo fractional derivative is:*

$$ {}^{C}D^{\alpha}_{a+}g(t) = I^{n-\alpha}_{a+}D^{n}_{a+}g(t), $$

such that:

$$ \beta > n \quad and \quad (n-1 \leq \alpha < n). $$

$$ {}^{C}D^{\alpha}_{a+}(t-a)^{\beta} \equiv I^{n-\alpha}_{a+}D^{n}_{a+}(t-a)^{\beta}. $$

$$ D^{\alpha}_{a+}(t-a)^{\beta} = \frac{d^{n}(x-a)^{\beta}}{dt^{n}} = \beta(\beta-1)\cdots(\beta-n+1)(t-a)^{\beta-n}. $$

$$ = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}(t-a)^{\beta-n}. $$

Substituting the result (1.38) into equation (1.36) and using relation (1.17), we obtain:

$$ {}^{C}D^{\alpha}_{a+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}(t-a)^{\beta-\alpha}. $$

Thus, we conclude:

$$ {}^{C}D^{\alpha}_{a+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}(t-a)^{\beta-\alpha}. $$

Example 1.5 *The fractional derivative of a constant* ($k \in \mathbb{R}$) *is zero:*

$$ {}^{C}D^{\alpha}_{a+}k = 0 \quad for \quad (k \in \mathbb{R}, \alpha > 0). $$

Lemma 1.1 *(Linearity). Let* $n-1 < \alpha < n$, $n \in \mathbb{N}$, *and let* f *and* g *be two functions such that* ${}^{C}D^{\alpha}_{a+}f(t)$ *and* ${}^{C}D^{\alpha}_{a+}g(t)$ *exist. Then,*

Proof. Since:

$$ {}^{C}D^{\alpha}_{a+}f(t) = I^{n-\alpha}_{a+}D^{n}f(t), $$

we obtain:

$$ \begin{aligned} {}^{C}D^{\alpha}_{a+}(\lambda f(t) + \gamma g(t)) &= I^{n-\alpha}_{a+}D^{n}[\lambda f(t) + \gamma g(t)] \\ &= \lambda I^{n-\alpha}_{a+}D^{n}f(t) + \gamma I^{n-\alpha}_{a+}D^{n}g(t) \\ &= \lambda {}^{C}D^{\alpha}_{a+}f(t) + \gamma {}^{C}D^{\alpha}_{a+}g(t). \end{aligned} $$

Lemma 1.2 *(Non-commutativity). Suppose* $n-1 < \alpha < n$, $n, m \in \mathbb{N}, \alpha \in \mathbb{R}_+$, *and* ${}^C D^{\alpha}_{a+} g(t)$ *exists. Then,*

$$ {}^C D^{\alpha}_{a+} D^{m}_{a+} g(t) - {}^C D^{m}_{a+} D^{\alpha}_{a+} g(t) \neq 0. $$

Remark. The fractional derivative in the sense of Riemann-Liouville (R-L) is also non-commutative.

Lemma 1.3 *The Leibniz rule for fractional derivatives. Let* f *and* g *be two functions in* $C^1[a,b]$*, then the Leibniz rule for fractional differentiation is given as:*

$$ D^{\alpha}_{+}(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} f^{(k)} D^{\alpha-k}_{+} g(t). $$

1.6 Fractional Derivatives of Some Functions

This section presents examples for the fractional derivative of constant functions, exponential functions, power functions, and trigonometric functions (sine and cosine).

1.6.1 The Constant Function

The fractional derivative of a constant is zero in the physical sense, but for $R-L$ we have:

$$ D^{\alpha}_{a+} k = \frac{k}{\Gamma(1-\alpha)} (t-a)^{-\alpha}, \quad k \neq 0, \quad k = \text{const}. $$

Lemma 1.4 *The fractional derivative of a constant* k *in the sense of Caputo is zero:*

$$ {}^C D^{\alpha}_{a+} k = 0, \quad k = const. $$

1.6.2 The Power Function

The Riemann-Liouville fractional derivative of order $\alpha > 0$ with $n-1 < \alpha < n$ for a power function $g(t) = t^p$ for $p \geq 0$ is given by:

$$ D^{\alpha}_{a+} t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, $$

$$ {}^C D^{\alpha}_{a+} t^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, & p > n-1, \\ 0, & p \leq n-1. \end{cases} $$

Example 1.6 *We compute the fractional derivative of* $g(t) = t^2$ *for* $0 < \alpha < 1$ *using formula (1.42) with* $\alpha \in \mathbb{R}_+$ *and* $\alpha \notin \mathbb{N}$*. The Caputo fractional derivative of the power function* $g(t)$ *is:*

$$ {}^{C}D_{a+}^{\alpha} t^2 = \frac{\Gamma(2+1)}{\Gamma(2-\alpha+1)} t^{2-\alpha} = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}, \quad n-1 < \alpha < n < 3 $$

For specific values of α:
- If $\alpha = \frac{1}{3}$:

$$ {}^{C}D_{a+}^{\frac{1}{3}} t^2 = \frac{2}{\Gamma(3-\frac{1}{3})} t^{\frac{5}{3}} = \frac{2}{\Gamma(\frac{8}{3})} t^{\frac{5}{3}} \approx 1.33 t^{\frac{5}{3}} $$

- If $\alpha = \frac{1}{2}$:

$$ {}^{C}D_{a+}^{\frac{1}{2}} t^2 = \frac{2}{\Gamma(3-\frac{1}{2})} t^{\frac{3}{2}} = \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} \approx 1.5 t^{\frac{3}{2}} $$

- If $\alpha = \frac{3}{4}$:

$$ {}^{C}D_{a+}^{\frac{3}{4}} t^2 = \frac{2}{\Gamma(3-\frac{3}{4})} t^{\frac{5}{4}} = \frac{2}{\Gamma(\frac{9}{4})} t^{\frac{5}{4}} \approx 1.77 t^{\frac{5}{4}} $$

Thus, the general formula is:

$$ {}^{C}D_{a+}^{\alpha} t^2 = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}, \quad t_{t,\alpha\to 0}^{2,\alpha-1} $$

1.6.3 Exponential Function

For the function $g(t) = e^{\lambda t}$, applying the Caputo operator gives:

$$ {}^{C}D_{a+}^{\alpha} e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = \lambda^n t^{n-\alpha} E_{1,n-\alpha+1}(\lambda t). $$

1.6.4 Sine and Cosine Functions

The behavior of the fractional Caputo derivatives for sine and cosine functions is given by:

Theorem 1.2 *Let $\lambda \in \mathbb{C}, \alpha \in \mathbb{R}_+, n \in \mathbb{N}, n-1 < \alpha < n$. Then,*

$$ {}^{C}D_{a+}^{\alpha} \sin(\lambda t) = -\frac{1}{2}(i\lambda)^n t^{n-\alpha} (E_{1,n-\alpha+1}(i\lambda t) - (-1)^n E_{1,n-\alpha+1}(-i\lambda t)). $$

Proof: First, we use the formula:

$$ \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}. $$

Applying formula (3.4) for the exponential function and the linearity property of the Caputo fractional derivative, we show that:

$$
\begin{aligned}
{}^C D_{a+}^{\alpha} \sin \lambda t &= {}^C D_{a+}^{\alpha} \frac{e^{i\lambda t} - e^{-i\lambda t}}{2i} \\
&= \frac{1}{2i} \left({}^C D_{a+}^{\alpha} e^{i\lambda t} - {}^C D_{a+}^{\alpha} e^{-i\lambda t} \right) \\
&= \frac{1}{2i} \left((i\lambda)^{n-\alpha} E_{1,n-\alpha+1}(i\lambda t) - (-i\lambda)^{n-\alpha} E_{1,n-\alpha+1}(-i\lambda t) \right) \\
&= \frac{-1}{2i} \left(i\lambda \right)^{n-\alpha} E_{1,n-\alpha+1}(i\lambda t) + (-1)^n E_{1,n-\alpha+1}(-i\lambda t) \big) .
\end{aligned}
$$

Theorem 1.3 *Let $\lambda \in \mathbb{C}$,$\alpha \in \mathbb{R}_+$, $n \in \mathbb{N}$, and $n-1 < \alpha < n$. Then:*

$$
{}^C D_{a+}^{\alpha} \cos \lambda t = \frac{1}{2} \left((i\lambda)^{n-\alpha} E_{1,n-\alpha+1}(i\lambda t) + (-1)^n E_{1,n-\alpha+1}(-i\lambda t) \right) .
$$

Proof Using:

$$
\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C}.
$$

$$
\begin{aligned}
{}^C D_{a+}^{\alpha} \cos \lambda t &= {}^C D_{a+}^{\alpha} \frac{e^{i\lambda t} + e^{-i\lambda t}}{2} \\
&= \frac{1}{2} \left({}^C D_{a+}^{\alpha} e^{i\lambda t} + {}^C D_{a+}^{\alpha} e^{-i\lambda t} \right) \\
&= \frac{1}{2} \left((i\lambda)^{n-\alpha} E_{1,n-\alpha+1}(i\lambda t) + (-i\lambda)^{n-\alpha} E_{1,n-\alpha+1}(-i\lambda t) \right) \\
&= \frac{1}{2} \left((i\lambda)^{n-\alpha} E_{1,n-\alpha+1}(i\lambda t) + (-1)^n E_{1,n-\alpha+1}(-i\lambda t) \right) .
\end{aligned}
$$

1.7 Hadamard fractional integral and derivative

Definition 1.7 *(Hadamard fractional integral). (see [24])*
The Hadamard fractional integral of order $\alpha > 0$ for a function $h : [1, +\infty) \to \mathbb{R}$ is defined as

$$
I_{a+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{\alpha-1} h(s) \frac{ds}{s}
$$

where Γ is the Gamma function.

Definition 1.8 *(Hadamard fractional derivative). (see [24])*
For a function h given on the interval $[1, +\infty)$, and $n-1 < \alpha < n$, the Hadamard derivative of order α is defined by

$$
D_{a+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt} \right)^n \int_a^t \left(\log \frac{t}{s} \right)^{n-\alpha-1} h(s) \frac{ds}{s} = \delta^n I_{a+}^{n-\alpha} h(t),
$$

where $n = \lfloor \alpha \rfloor + 1$, *and* $\lfloor \alpha \rfloor$ *denotes the integer part of the real number* α *and* $\delta = t\frac{d}{dt}$, *provided the right integral converges.*

There is a recent generalization introduced by Jarad and ali in [22], where the authors define the generalization of the Hadamard fractional derivatives and present properties of such derivatives. This new generalization is now known as the Caputo-Hadamard fractional derivatives and is given by the following definition:

Definition 1.9 *(Caputo-Hadamard fractional derivative). (see [22]).*
Let $a = 0$, *and* $n = \lfloor \alpha \rfloor + 1$. *If* $h(t) \in AC^n_\delta[a,b]$, *where* $0 < a < b < \infty$ *and*

$$AC^n_\delta[a,b] = \left\{ h : [a,b] \to \mathbb{C} : \delta^{n-1} h \in AC_\delta[a,b] \right\}.$$

The left-sided Caputo-type modification of left-Hadamard fractional derivatives of order α *is given by*

$${}^C_H D^\alpha_{a+} h(t) = D^\alpha_{a+} \left(h(t) - \sum_{k=0}^{n-1} \frac{\delta^k h(a)}{k!} \left(\log \frac{t}{s} \right)^k \right)$$

Theorem 1.4 *(See [22])*
Let $\alpha > 0$, *and* $n = \lfloor \alpha \rfloor + 1$. *If* $y(t) \in AC^n_\delta[a,b]$, *where* $0 < a < b < \infty$. *Then*
${}^C_H D^\alpha_{a+} f(t)$ *exist everywhere on* $[a,b]$ *and*
(i) if $\alpha \notin \mathbb{N} - \{0\}$, ${}^C_H D^\alpha_{a+} f(t)$ *can be represented by*

$${}^C_H D^\alpha_{a+} h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{n-\alpha-1} \delta^n h(s) \frac{ds}{s} = I^{n-\alpha}_{a+} \delta^n h(t),$$

(ii) if $\alpha \in \mathbb{N} - \{0\}$, *then*

$${}^C_H D^\alpha_{a+} h(t) = \delta^n h(t)$$

In particular

$${}^C_H D^0_{a+} h(t) = h(t)$$

Caputo-Hadamard fractional derivatives can also be defined on the positive half axis $\mathbb{R}^+$ *by replacing a by 0 in formula (2.4) provided that* $h(t) \in AC^n_{\mathcal{H}}(\mathbb{R}^+)$. *Thus one has*

$${}^C_H D^\alpha_{a+} h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{n-\alpha-1} s^n h^{(n)}(s) \frac{ds}{s}$$

Proposition 1.1 *(see [24]).*
Let $\alpha > 0, \beta > 0, n = \lfloor \alpha \rfloor + 1, anda > 0$, *then*

$$I^\alpha_{a+} (\log \tfrac{1}{x})^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\log \tfrac{1}{x})^{\beta+\alpha-1}$$

$${}^C_H D^\alpha_{a+} (\log \tfrac{1}{x})^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\log \tfrac{1}{x})^{\beta-\alpha-1},\ \beta > n,$$

$${}^C_H D^\alpha_{a+} (\log \tfrac{1}{x})^k = 0,\ for\ k = 0, 1, ..., n-1.$$

Theorem 1.5 *(see [20])*
Let u(t)$\in AC^n_{\mathcal{H}}[a,b]$, $0 < a < b < \infty$ *and* $\alpha \geq 0$, $\beta \geq 0$, *Then*

$$
{}^{C}_{H}D^{\alpha}_{a+}(I^{\alpha}_{a+}u)(t) = (I^{\beta-\alpha}_{a+}u)(t),
$$
$$
{}^{C}_{H}D^{\alpha}_{a+}\left({}^{C}_{H}D^{\beta}_{a^+}u\right)(t) = \left({}^{C}_{H}D^{\alpha+\beta}_{a^+}u\right)(t).
$$

Example 1.7 *Caputo-Hadamard Derivative of* $f(t) = \ln t$ *Let* $\alpha = 0.5$, $n = 1$*:*

$$
{}^{CH}D^{0.5}\ln t = \frac{1}{\Gamma(0.5)}\int_1^t \left(\ln\frac{t}{\tau}\right)^{-0.5}\left(\tau\frac{d}{d\tau}\ln\tau\right)\frac{d\tau}{\tau}.
$$

Since $\tau\frac{d}{d\tau}\ln\tau = 1$, *this simplifies to:*

$$
= \frac{1}{\sqrt{\pi}}\int_1^t \left(\ln\frac{t}{\tau}\right)^{-0.5}\frac{d\tau}{\tau}.
$$

Let $u = \ln\frac{t}{\tau}$, *then:*

$$
= \frac{1}{\sqrt{\pi}}\int_0^{\ln t} u^{-0.5}\,du = \frac{2\sqrt{\ln t}}{\sqrt{\pi}}.
$$

2. Comparison of Caputo, Hadamard, and Riemann-Liouville Derivatives

Property	Caputo D^{α}	Hadamard $\mathcal{H}^{\alpha}$	R-L D^{α}
Kernel	$(t-\tau)^{n-\alpha-1}$	$\left(\ln\frac{t}{\tau}\right)^{\alpha-1}$	$(t-\tau)^{n-\alpha-1}$
Initial Cond.	Uses $f^{(n)}(0)$	Logarithmic scaling	Fractional initial
Example ($f(t) = t$)	$\frac{2\sqrt{t}}{\sqrt{\pi}}$	Complex	$\frac{t^{-0.5}}{\Gamma(0.5)}$
Example ($f(t) = \ln t$)	Not standard	$\frac{(\ln t)^{-0.5}}{\Gamma(0.5)}$	Not standard
Applications	Physics, engineering	Fractal geometry, economics	Pure mathematic

Some Fixed Point Theorems 2

2.1 Spaces of integrable, absolutely continuous, and continuous functions

In this section we present different functional spaces which are used later.

Let $I = [a, b]$, $0 < a < b$, be a finite or infinite interval of the real positive interval $[0, \infty)$ and E be a real Banach space with the norm $\| \cdot \|$. We denote by $L^1(I, \mathbb{R})$ the space of Lebesgue integrable functions h on I with the norm

$$\|h\|_{L^1} = \int_a^b |h(t)| dt.$$

A measurable function $h : I \longrightarrow E$ is Bochner integrable if and only if $\|h\|$ is Lebesgue integrable. Let $L^1(I, \mathbb{R})$ be the space of E-valued Bochner integrable functions on I with the norm

$$\|h\|_{L^1} = \int_a^b ||h(t)|| dt.$$

Let the space

$$AC^n_\delta([a, b], E) = \{h : [a, b] \to E : \delta^{n-1} h(t) \in AC([a, b]), E)\}$$

were $\delta = t\frac{d}{dt}$ is the Hadamard derivative (see Definition **??** bellow) and $AC([a, b], E)$ is the space of absolutely continuous functions on $[a, b]$.

Definition 2.1 *Let g be a function from a set E to itself. A fixed point of g is any point $x \in E$ such that:*

$$g(x) = x.$$

Definition 2.2 *A subset A of $(E, \| \cdot \|)$ is said to be relatively compact if its closure is compact.*

Definition 2.3 *Let E and G be two Banach spaces, and let $g : E \to G$. The function g is called completely continuous if it is continuous and maps every bounded subset of E to a relatively compact subset of G. In this case, g is compact if $g(E)$ is relatively compact in G.*

Definition 2.4 *Let $A \subset (J, \mathbb{R})$. The set A is said to be equicontinuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that:*

$$|t_2 - t_1| < \delta \Rightarrow |g(t_1) - g(t_2)| \leq \varepsilon$$

for all $t_1, t_2 \in J$ and all $g \in A$.

Definition 2.5 *Let $(E, \|\cdot\|)$ be a Banach space. A function $g : E \to E$ is said to be Lipschitz with constant $k > 0$ if:*

$$\forall x, y \in E, \quad \|g(x) - g(y)\| \leq k\|x - y\|.$$

If $0 < k < 1$, then g is called a contraction.

2.2 Some Fixed Point Theorems

We are interested in Banach's fixed-point theorem, which ensures the existence and uniqueness of a fixed point. We also discuss the fixed-point theorem of Schaefer, Schauder's theorem, the nonlinear alternative of Leray-Schauder, and Krasnoselskii's fixed-point theorem, which guarantees only the existence of a fixed point.

2.2.1 Banach Fixed-Point Theorem

Banach's fixed-point theorem, also known as the contraction mapping theorem, is a fundamental result that guarantees the existence of a unique fixed point for any contraction mapping in a metric space.

Theorem 2.1 *Let $(E, \|\cdot\|)$ be a Banach space and $A : E \to E$ a contraction. Then, the operator A has a unique fixed point $x \in E$. Moreover, if $x_0 \in E$ and $x_n = Ax_{n-1}$, then:*

$$x = \lim_{n\to\infty} x_n.$$

Proof Let k be the contraction constant of A and x_0 an element of E. Define the sequence $\{x_n\}$ in E by:

$$x_n = Ax_{n-1}, \quad \forall n \geq 1.$$

Since A is a contracting operator, we obtain:

$$\|x_n - x_{n+1}\| = \|Ax_{n-1} - Ax_n\| \leq k\|x_{n-1} - x_n\|, \quad \forall n \geq 1.$$

Thus,

$$\|x_n - x_{n-1}\| \leq k^n\|x_0 - x_1\|, \quad \forall n \geq 1.$$

Consequently, for any $m > n$, we have:

$$\|x_n - x_m\| \leq (k^n + k^{n+1} + \cdots + k^{m-1})\|x_0 - x_1\| \leq \frac{k^n}{1-k}\|x_0 - x_1\|.$$

This shows that $\{x_n\}$ is a Cauchy sequence in E, which is a Banach space, so $x_n \to x$ for some $x \in E$. By the continuity of A, we get:

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_{n-1} = Ax.$$

To show the uniqueness of the fixed point in E, suppose that x and y are two fixed points of A. Then,

$$\|x - y\| = \|A(x) - A(y)\| \le k\|x - y\| \le \|x - y\|,$$

which implies $x = y$.

2.2.2 Fixed Point Theorem of Schaefer

Theorem 2.2 *Let X be a Banach space. If $A : X \to X$ is a completely continuous operator and the set*

$$\varepsilon = \{x \in X : \lambda Ax = x \text{ for some } \lambda \in [0,1]\}$$

is bounded, then A has at least one fixed point.

2.2.3 Schauder Fixed Point Theorem

The Schauder fixed point theorem proves the existence of a fixed point for continuous functions on a convex compact set in a Banach space and states that a continuous application admits a fixed point, which is not necessarily unique.

Theorem 2.3 *Let (E, d) be a complete metric space, let X be a convex and closed subset of E, and let $A : X \to X$ be a mapping such that the set $\{Ax : x \in X\}$ is relatively compact in E. Then A has at least one fixed point.*

2.2.4 Nonlinear Alternative of Leray-Schauder

Theorem 2.4 *Let X be a Banach space, Ω an open bounded subset of X, with $0 \in \Omega$, and let $A : \bar{\Omega} \to X$ be a compact mapping, then:*

1. *A has a fixed point in $\bar{\Omega}$, or*
2. *There exists $\lambda \in (0,1)$ and $x \in \partial\Omega$ such that: $x = \lambda A(x)$.*

2.2.5 Arzelà-Ascoli Theorem

Theorem 2.5 *Let A be a subset of $C(J, E)$. A is relatively compact in $C(J, E)$ if and only if the following conditions are satisfied:*

(i) The set A is uniformly bounded, i.e., there exists a constant $k > 0$ such that $\|f(x)\| \le k$, for all $x \in J$ and all $f \in A$.

(ii) The set A is equicontinuous, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|t_1 - t_2| < \delta \Rightarrow \|f(t_1) - f(t_2)\| \leq \varepsilon,$$

for all $t_1, t_2 \in J$ and all $f \in A$.

2.2.6 Krasnoselskii's Theorem

Theorem 2.6 *Let X be a Banach space and let M be a non-empty, convex, and closed subset of X. Suppose A and B are two operators from X into X satisfying:*

(i) $Ax + By \in M$, for all $x, y \in M$.

(ii) A is a contraction.

(iii) B is completely continuous.

Then there exists $x^* \in M$ such that:

$$Ax^* + Bx^* = x^*.$$

Existence Results for a Nonlinear Caputo-Hadamard Fractional Differential Equation with Hadamard Integral and Anti-Periodic Conditions 3

This chapter essentially contains the paper [11] *"Boundary Value Problem for Nonlinear Caputo-Hadamard Fractional Differential Equation with Hadamard Fractional Integral and Anti-Periodic Conditions*

Abstract. In this chapter, we investigates a class of boundary value problems for nonlinear fractional differential equations involving the Caputo-Hadamard derivative and Hadamard fractional integral. We establish sufficient conditions ensuring the existence and uniqueness of solutions under anti-periodic boundary conditions. Our analysis combines tools from fractional calculus and fixed-point theory to derive rigorous qualitative results.
.

3.1 Introduction and Motivations

The aim of this chapter is to study a class of boundary value problems involving a fractional-order differential equation with the Caputo-Hadamard fractional derivative. Sufficient conditions will be established to guarantee the existence and uniqueness of solutions for this fractional boundary value problem. The boundary conditions considered in this work are of a very general nature and can be reduced to many special cases by adjusting the parameters involved.

B. Ahmad and J. J. Nieto investigated the existence and uniqueness of solutions for anti-periodic fractional boundary-value problem

$$^{c}D^{\alpha}x(t) = f(t, x(t)), \quad t \in [0, T], T > 0, 1 < \alpha \leq 2$$

$$x(0) = -x_T, ^{c}D^{\alpha}x(0) = -^{c}D^{\alpha}x(T), 1 < \alpha < 2$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative of order α and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Some very recent studies on fractional equations with Caputo derivative and Riemann–Liouville derivatives are [5]. The authors in [12] investigated the existence and uniqueness of solutions for the BVP

$$^{c}D^{\nu}y(t) = f(t, y(t), ^{c}D^{\nu}y(t)), \text{ for each, } t \in J := [0, T], \ T > 0, \ 0 < \nu \leq 1,$$

$$ay(0) + by(T) = c,$$

where $^{c}D^{\nu}$ is the Caputo fractional derivative, $(E, ||\cdot||)$ is a real Banach space, $f : J \times E \times E \to E$ is a given function and a, b are real with $a + b \neq 0$ and $c \in E$. and

$$^{c}D^{\nu}y(t) = f(t, y(t), ^{c}D^{\nu}y(t)), \ \text{ for every } t \in J := [0, T], T > 0, \ \ 0 < \nu \leq 1$$

$$y(0) + g(y) = y_0$$

where $^{c}D^{\nu}$ is the Caputo fractional derivative, $(E, ||\cdot||)$ is a real Banach space, $f : J \times E \times E \to E$ is a given function, $g : C(J, E) \to E$ is a continuous function and $y_0 \in E$.

Motivated by aforesaid works, in this chapitre, we introduce and investigate the following nonlinear multi-term fractional order boundary value problem with nonlocal integral conditions:

$$^{C}_{H}D^{\alpha}x(t) = f(t, x(t)), \quad 0 < \alpha < 1, \ 1 < t < T, \tag{3.1}$$

$$ax(1) + bx(T) = \lambda I^{\sigma}x(s)ds, \quad 0 < \sigma < 1, \ \lambda \in \mathbb{R}, \tag{3.2}$$

where $^{C}_{H}D^{\alpha}$ denote the Caputo-Hadamard fractional derivative of order α , $0 < \alpha < 1$ and $f : I \times E \to E$ is a given continuous functions. Here, E is a Banach space with norm $\|\cdot\|$ and $I = [1, T]$, a,b ,λ are real constants and $1 < s < T$.

More exactly, we prove the existence of solutions for the above problem using Schauder's fixed point theorem nonlinear alternative for single valued maps, and Scheafer's fixed point theorem are given. Also, we preve the uniqueness results by means of Boyd and Wong's and Banach's fixed point theorems. The obtained result is illustrated by an example.

3.2 Preliminaries and lemmas

At first, we recall some concepts on fractional calculus and present some additional properties that will be used later. For more details, we refer to . We present some basic definitions and results from fractional calculus theory.

Let $E = C([1,T],\mathbb{R})$ be the Banach space of all continuous functions from $[1,T]$ into $\mathbb{R}$ with the norm

$$\|u\| = \max_{t\in[1,T]} |u(t)|.$$

Let the space

$$AC^{\delta}([a,b],\mathbb{R}) = \{h : [a,b] \to \mathbb{R} : \delta^{n-1}h(x) \in AC([a,b],\mathbb{R})\},$$

where $\delta = t\frac{d}{dt}$ is the Hadamard derivative and $AC([a,b],\mathbb{R})$ is the space of absolutely continuous functions on $[a,b]$.

Lemma 3.1 *(see [22]).*
Let $\alpha \geq 0$, and $n = \lfloor \alpha \rfloor + 1$. If $u(t) \in AC^{n}_{\delta}[a,b]$, then the Caputo-Hadamard fractional differential equation

$${}^{C}_{H}D^{\alpha}_{a^+}u(t) = 0,$$

has a solution:

$$u(t) = \sum_{k=0}^{n-1} c_k \left(\log\frac{t}{a}\right)^k,$$

and the following formula holds:

$$I^{\alpha}_{a+}\left({}^{C}_{H}D^{\alpha}_{a^+}u\right)(t) = u(t) + \sum_{k=0}^{n-1} c_k \left(\log\frac{t}{a}\right)^k,$$

where $c_k \in \mathbb{R}, k = 1,2,...,n-1$.

3.3 Existence of Solutions Results

First, we prove a preparatory lemma for boundary value problem of linear fractional differential equations with Caputo-Hadamard derivative.

Definition 3.1 *A function $x(t) \in AC^{1}_{\delta}(J,\mathbb{R})$ is said to be a solution of* (3.1), (3.2) *if x satisfies the equation ${}^{C}_{H}D^{\alpha}x(t) = f(t,x(t)) on J$, and the conditions* (3.2).

For the existence of solutions for the problem (1:1); (1:2), we need the following auxiliary lemma.

Lemma 3.2 *Let $h : [1,+\infty) \to \mathbb{R}$ be a continuous function. A function x is a solution of the fractional integral equation*

$$x(t) = I^{r}h(t) + \frac{1}{\Lambda}\left\{\lambda I^{r+q}h(\eta) - \beta I^{r}h(T) + \delta\right\}$$

if and only if x is a solution of the fractional BVP

$$\begin{aligned} &{}^{C}_{h}D^{r}x(t) = h(t), \quad t \in J, \quad r \in (0,1] \\ &\alpha x(1) + \beta x(T) = \lambda I^{q}x(\eta) + \delta, \quad q \in (0,1] \end{aligned}$$

Proof. Assume x satisfies (**??**). Usuing then Lemma 2.7 implies that

$$x(t) = I^r h(t) + c_1.$$

By applying the boundary conditions (**??**) in (3.3), we obtain

$$\alpha c_1 + \beta I^r h(T) + \beta c_1 = \lambda I^{r+q} h(\eta) + c_1 \frac{\lambda (\log \eta)^q}{\Gamma(q+1)} + \delta.$$

Thus,

$$c_1 \left(\alpha + \beta - \frac{\lambda (\log \eta)^q}{\Gamma(q+1)} \right) = \lambda I^{r+q} h(\eta) - \beta I^r h(T) + \delta.$$

Consequently,

$$c_1 = \frac{1}{\Lambda} \left\{ \lambda I^{r+q} h(\eta) - \beta I^r h(T) + \delta \right\},$$

where

$$\Lambda = \left(\alpha + \beta - \frac{\lambda (\log \eta)^q}{\Gamma(q+1)} \right).$$

Finally, we obtain the solution (3.2):

$$x(t) = I^r h(t) + \frac{1}{\Lambda} \left\{ \lambda I^{r+q} h(\eta) - \beta I^r h(T) + \delta \right\}.$$

□

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (1.1), (1.2) by using a variety of fixed point theorems.

3.4 Existence and uniqueness result via Banach's fixed point theorem:

Theorem 3.1 *Assume the following hypothesis:*
(H1) *There exists a constant* $L > 0$ *such that*

$$|f(t,x) - f(t,y)| \leq L|x-y|.$$

If

$$LM < 1,$$

with

$$M := \left\{ \frac{(\log T)^\tau}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^\tau}{|\Lambda|\Gamma(r+1)} \right\},$$

then the problem (1.1) has a unique solution on J.

Proof. Transform the problem (1.1), (1.2) into a fixed point problem for the operator $\mathfrak{F}$ defined by

$$Fx(t) = \Gamma^r h(t) + \frac{1}{\Lambda}\left\{\Lambda T^{r+q} h(\eta) - \beta \Gamma^r h(T) + \delta\right\}.$$

Applying the Banach contraction mapping principle, we shall show that F is a contraction.

Now let $x, y \in C(J, \mathbb{R})$. Then, for $t \in J$, we have

$$\|(Fx)(t) - (Fy)(t)\|_\infty \leq LM\|x - y\|_\infty.$$

Thus, we deduce that F is a contraction mapping. As a consequence of the Banach contraction principle, the problem (1.1)-(1.2) has a unique solution on J. This completes the proof. □

3.5 Existence result via Schaefer's fixed point theorem:

Theorem 3.2 *Assume the hypotheses:*
(H2): *The function $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.*
(H3) *There exists a constant $K > 0$, such that*

$$|f(t, 0)| \leq K, \quad \textit{for a.e. } t \in J.$$

Then, the problem (1.1)-(1.2) has at least one solution in J.

Proof. We shall use Schaefer's fixed point theorem to prove that $\mathfrak{F}$ defined by (3.6) has a fixed point. The proof will be given in several steps.

Step 1: $\mathfrak{F}$ is continuous. Let x_n be a sequence such that $x_n \to x$ in $C(J, \mathbb{R})$. Then for any $r > 0$, we take

$$\begin{aligned}
\|(\mathfrak{F}x_n)(t) - (\mathfrak{F}x)(t)\| \leq & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \|f(s, x_n(s)) - f(s, x(s))\| \frac{ds}{s} \\
& + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{r+q-1} \|f(s, x_n(s)) - f(s, x(s))\| \frac{ds}{s} \\
& + \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \|f(s, x_n(s)) - f(s, x(s))\| \frac{ds}{s} \\
\leq & \left\{\frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\} \|f(s, x_n(s)) - f(s, x(s))\|.
\end{aligned}$$

Since f is continuous, we have $\|(\mathfrak{F}x_n)(t) - (\mathfrak{F}x)(t)\|_\infty \to 0$ as $n \to \infty$.

Step 2: $\mathfrak{F}$ maps bounded sets into bounded sets in $C(J, \mathbb{R})$

Indeed, it is enough to show that for any $r > 0$, we take

$$u \in B_r = \{x \in C(J, \mathbb{R}), \|x\|_\infty \leq r\}.$$

From (H1) and (H3), we have

$$|f(s,x(s))| \leq |f(s,x(s)) - f(t,0)| + |f(t,0)| \leq Lr + K.$$

For $x \in B_r$ and for each $t \in [1,T]$, we have

$$\begin{aligned}
|(\mathfrak{F}x)(t)| &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |f(s,x(s))| \frac{ds}{s} \\
&\quad + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{r+q-1} |f(s,x(s))| \frac{ds}{s} \\
&\quad + \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |f(s,x(s))| \frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\
&\leq \frac{Lr+K}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{ds}{s} \\
&\quad + \frac{|\lambda|(Lr+K)}{|\Lambda|\Gamma(r+q)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{r+q-1} \frac{ds}{s} \\
&\quad + \frac{|\beta|(Lr+K)}{|\Lambda|\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\
&\leq (Lr+K) \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\} + \frac{|\delta|}{|\Lambda|} \\
&\leq (Lr+K)M + \frac{|\delta|}{|\Lambda|}.
\end{aligned}$$

Thus,

$$\|(\mathfrak{F}x)(t)\| \leq (Lr+K)M + \frac{|\delta|}{|\Lambda|}.$$

Step 3: $\mathfrak{F}$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.

Let $t_1, t_2 \in J$, $t_1 < t_2$, B_r be a bounded set of $C(J,\mathbb{R})$ as in Step 2, and let $x \in B_r$. Then

$$\begin{aligned}
\|\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1)\| &\leq \frac{1}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] \|f(s,x(s))\| \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \|f(s,x(s))\| \frac{ds}{s} \\
&\leq \frac{Lr+K}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] \frac{ds}{s} \\
&\quad + \frac{K}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \frac{ds}{s} \\
&\leq \frac{Lr+K}{\Gamma(r+1)} \left[(\log t_2)^r - (\log t_1)^r \right],
\end{aligned}$$

which implies $\|\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1)\|_\infty \to 0$ as $t_1 \to t_2$.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $\mathfrak{F}$ is continuous and completely continuous.

Step 4: A priori bounds.

Now it remains to show that the set

Consider the set

$$\Lambda = \{x \in C(J, \mathbb{R}) : x = \rho\mathfrak{F}(x) \text{ for some } 0 < \rho < 1\}.$$

Proof of boundedness. For any $x \in \Lambda$ and each $t \in J$, we have:

$$\begin{aligned} x(t) \leq \rho \Bigg\{ & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} f(s, x(s)) \frac{ds}{s} \\ & + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{r+q-1} f(s, x(s)) \frac{ds}{s} \\ & + \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} f(s, x(s)) \frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \Bigg\} \end{aligned}$$

For $\rho \in [0, 1]$, let x satisfy for each $t \in J$:

$$\begin{aligned} \|\mathfrak{F}x(t)\| \leq & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |f(s, x(s))| \frac{ds}{s} \\ & + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{r+q-1} |f(s, x(s))| \frac{ds}{s} \\ & + \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |f(s, x(s))| \frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\ \leq & (Lr + K)M + \frac{|\delta|}{|\Lambda|}. \end{aligned}$$

Thus

$$\|\mathfrak{F}x(t)\| \leq \infty$$

This implies that the set Λ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $\mathfrak{F}$ has a fixed point which is a solution on J of the problem (1.1)-(1.2). □

3.6 Existence via the Leray–Schauder nonlinear alternative

Theorem 3.3 *Assume the following hypotheses:*

(H4) There exist $\omega \in L^1(J, \mathbb{R}^+)$ *and* $\psi : [0, \infty) \to (0, \infty)$ *continuous and nondecreasing such that*

$$|f(t, x)| \leq \omega(t)\psi(\|x\|), \textit{ for a.e. } t \in J \textit{ and each } x \in \mathbb{R}.$$

(H5) There exists a constant $\epsilon > 0$ such that

$$\frac{\epsilon}{\|\omega\|\psi(\epsilon)M + \frac{|\delta|}{|A|}} > 1.$$

Then the boundary value problem (1.1)-(1.2) has at least one solution on J.

Proof. We shall use the Leray–Schauder theorem to prove that $\mathfrak{F}$ defined by (3.6) has a fixed point. As shown in Theorem 3.4, we see that the operator $\mathfrak{F}$ is continuous, uniformly bounded, and maps bounded sets into equicontinuous sets. So by the Arzelà–Ascoli theorem $\mathfrak{F}$ is completely continuous.

Let x be such that for each $t \in J$, we take the equation $x = \lambda m x$ for $\lambda \in (0,1)$ and let x be a solution. After that, the following is obtained:

$$\begin{aligned}
|x(t)| &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \omega(t)\psi(\|x\|)\frac{ds}{s} \\
&\quad + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{r+q-1} \omega(t)\psi(\|x\|)\frac{ds}{s} \\
&\quad + \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \omega(t)\psi(\|x\|)\frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\
&\leq \|\omega\|\psi(\|x\|)M + \frac{|\delta|}{|\Lambda|}.
\end{aligned}$$

□

and consequently

$$\frac{\|x\|_\infty}{\|\omega\|\psi(\|x\|)M + \frac{|\delta|}{|\Lambda|}} \leq 1.$$

Then by condition (H5), there exists ϵ such that $\|x\|_\infty \neq \epsilon$. Let us set

$$\kappa = \{x \in C(J, \mathbb{R}) : \|x\| < \epsilon\}.$$

Obviously, the operator $\text{Im} : \overline{\kappa} \to C(J, \mathbb{R})$ is completely continuous. From the choice of κ, there is no $x \in \partial\kappa$ such that $x = \lambda \text{Im}(x)$ for some $\lambda \in (0,1)$. As a result, by the Leray-Schauder's nonlinear alternative theorem, $\mathfrak{F}$ has a fixed point $x \in \kappa$ which is a solution of the (1.1)-(1.2).
The proof is completed. □

Now we present another variant of existence-uniqueness result.

3.7 Existence and uniqueness result via Boyd-Wong nonlinear contraction:

Definition 3.2 *Assume that E is a Banach space and $T : E \to E$ is a mapping. If there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property: $\|Tx - Ty\| \leq \psi(\|x - y\|), x, y \in E$, then, we say that T is a nonlinear contraction.*

Theorem 3.4 *(Boyd-Wong Contraction Principle)*
Suppose that B is a Banach space and $T : B \to B$ is a nonlinear contraction. Then T has a unique fixed point in B.

Theorem 3.5 *Assume that $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $H > 0$ satisfying the condition*

$$|f(t,x) - f(t,y)| \leq \frac{|x-y|}{H + |x-y|}, \quad \text{for } t \in J, x, y \in \mathbb{R}.$$

Then the fractional BVP (1.1)-(1.2) has a unique solution on J.

Proof. We define an operator $\mathfrak{F} : \chi \to \chi$ as in (3.6) and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(\varepsilon) = \frac{H\varepsilon}{H+\varepsilon}, \quad \forall \varepsilon > 0,$$

where $M \leq H$. We notice that the function ψ satisfies $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$. For any $x, y \in \chi$, and for each $t \in J$, we obtain

$$\begin{aligned}
|(\mathfrak{F}x)(t) - (\mathfrak{F}y)(t)| &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \|f(s,x(s)) - f(s,y(s))\| \frac{ds}{s} \\
&\quad + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{r+q-1} \|f(s,x(s)) - f(s,y(s))\| \frac{ds}{s} \\
&\quad + \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \|f(s,x(s)) - f(s,y(s))\| \frac{ds}{s} \\
&\leq \frac{|x-y|}{H+|x-y|} \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\} \\
&:= M \frac{|x-y|}{H+|x-y|} \\
&\leq \psi(\|x-y\|).
\end{aligned}$$

Then, we get $\|\mathfrak{F}x - \mathfrak{F}y\| \leq \psi(\|x-y\|)$. Hence, $\mathfrak{F}$ is a nonlinear contraction. Thus, by Theorem 3.9 (Boyd-Wong Contraction Principle) the operator $\mathfrak{F}$ has a unique fixed point which is the unique solution of the fractional BVP (1.1)-(1.2). The proof is completed.

3.8 Example

We consider the problem for Caputo-Hadamard fractional differential equations of the form:

$$\begin{cases} {}^{C}_{H}D^{\frac{2}{3}} x(t) = f(t, x(t)), & (t,x) \in ([1,e], \mathbb{R}^+), \\ x(1) + x(e) = \frac{1}{2}(I^{\frac{1}{2}} x(2) + \frac{3}{4}. \end{cases}$$

Here

$$r = \frac{2}{3}, \quad \delta = \frac{3}{4}, \quad q = \frac{1}{2}, \quad \lambda = \frac{1}{2}, \quad \alpha = 1, \quad \eta = 2, \quad \beta = 1, \quad T = e.$$

With

$$f(t, y(t)) = \frac{1}{t^2 + 4} \cos x, \quad t \in [1, e]$$

Clearly, the function f is continuous.
For each $x \in \mathbb{R}^+$ and $t \in [1, e]$, we have

$$|f(t, x(t)) - f(t, y(t))| \leq \frac{1}{4}|x - y|$$

Hence, the hypothesis (H1) is satisfied with $L = \frac{1}{4}$.
Further,

$$M := \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \simeq 2.0286$$

and

$$LM \simeq 0.5071 < 1.$$

Therefore, by the conclusion of Theorem 3.3, it follows that the problem (4.1) has a unique solution defined on $[1, e]$.

4 CONCLUSIONS

This thesis introduced the concept of Fractional Calculus; the branch of Mathematics which explores fractional integrals and derivatives. We first gave some basic techniques and functions, such as the Gamma function, the Beta function and the Mittag-Leffler function, which were necessary to understand the rest of this work. Thereafter we proved the construction of the Caputo and the Riemann Liouville method to define a differential. Therefore we used the forward difference derivative and the Cauchy formula for repeated integration respectively. We proved they are both linear and gave an expression for the Leibniz rule for fractional derivatives. We also explored the composition of fractional integrals and fractional derivatives. After giving the framework of differentials we were able to make use of it. We explored examples of some frequently used functions, namely the Power function, the Exponential function and the Trigonometric functions.

Next we studied Fractional Existence Results for a Nonlinear Caputo-Hadamard Fractional Differential Equation with Hadamard Integral and Anti-Periodic Conditions .

Bibliography

[1] R.P. Agarwal and B. Ahmad, Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions, *Comput. Math. Appl.* **62** (2011), 1200-1214.

[2] R.P Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Adv. Stud. Contemp. Math.* **16** (2) (2008), 181-196.

[3] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[4] B. Ahmad and J.J. Nieto, Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations. *Abstrac. Appl. Anal.* **2009** (2009), Article ID 494720, 9 pages.

[5] B. Ahmad and J.J. Nieto, Existence of solutions for impulsive anti-periodic boundary value problems of fractional order, *Taiwaness J. Math.*, **15** (3) (2011), 981-993.

[6] B. Ahmad and J.J. Nieto, Anti-periodic fractional boundary value problems, *Computers and Mathematics with Applications*, **62** (3) (2011), 1150-1156.

[7] B. Ahmad, J.J. Nieto, A. Alsaedi and N. Mohamad, On a New Class of Anti-periodic Fractional Boundary Value Problems, *Abst. and Applied Analysis*, (2013), 7 p.

[8] G. A. Anastassiou, *Advances on Fractional Inequalities*, Springer New York Dordrecht Heidelberg London, 2011.

[9] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems,* Kluwer Academin Publishers, Dordrecht (2003).

[10] D. Baleanu, J.A.T. Machado and A.C.-J. Luo *Fractional Dynamics and Control*, Springer, 2012.

[11] A. Boutiara, M. Benbachir and K. Guerbati *Boundary Value Problem for Nonlinear Caputo-Hadamard Fractional Differential Equation with Hadamard Fractional Integral and Anti-Periodic Conditions*, Mathematica · December 2021

[12] M. Benchohra, S.Bouriah and M.Darwish, Existence Results for Nonlinear BVP for Implicit Differential Equations of Fractional Order in Banach Space, *Fixed Point Theory*.

[13] M. Benchohra, S. Hamani and S.K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.

[14] T.A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii-Schaefer type. *Math. Nachr.* **189** (1989), 23-31.

[15] A. Chen and Y. Chen, Existence of Solutions to Anti-Periodic Boundary value problem for nonlinear Fractional Differential Equations with Impulses. *Adv. Difference. Equat.* (2011), 17 pages.

[16] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, 2010.

[17] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[18] Z. Guo, M. Liu, Existence and uniqueness of solutions for fractional order integrodifferential equations with nonlocal initial conditions. *Panamer. Math. J.* **21** (2011), 51-61.

[19] J. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, , Applied Mathematicals Sciences, 99, Springer-Verlag, New York, 1993.

[20] R. Hermann, *Fractional Calculus: An Introduction For Physicists*, World Scientific Publishing Co. Pte. Ltd. 2011.

[21] N. Heymans and I. Podlubny, Pysical interpretation of initial conditions for fractional differential equations with Riemnn-Liouville fractional derivatives, *Rheol Acta* **45** (2006), 765-771.

[22] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[23] M. Hu and L. Wang, Existence of solutions for a nonlinear fractional differential equation with integral boundary condition, *Int. J. Math. Comp. Sc.*, **7** (1) (2011).

[24] A.A. Kilbas and S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Diff. Equat.* **41** (2005), 84-89.

[25] A.A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[26] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.

[27] K.B. Oldham and J. Spanier, *The Fractional Calculus: theory and application of differentiation and integration to arbitrary order*, Academic Press, New York, London, 1974.

[28] M. D. Ortigueira, *Fractional Calculus for Scientists and Engineers*. Lecture Notes in Electrical Engineering, 84. Springer, Dordrecht, 2011.

[29] B. Ross, *Fractional Calculus and Its Applications*, Proceedings of the International Conference, New Haven, Springer-Verlag, New York (1974)

[30] I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.* **26** (2010), 103-107.

[31] X. Su and L. Liu, Existence of solution for boundary value problem of nonlinear fractional differential equation, *Appl. Math.* **22** (3) (2007) 291-298.

Abstract

In this work, we define some types of fractional derivatives, such as Riemann-Liouville, Caputo, and Hadamard, while presenting various properties related to these fractional derivatives. Then, we study the existence and uniqueness of solutions to certain problems represented by multi-order fractional differential equations, such as fractional boundary value problems (FBVP).

These problems involve fractional derivatives like Caputo-Hadamard, along with nonlocal boundary conditions in Banach spaces. The results obtained in this study are based on fixed-point theorems: the Banach contraction principle, Sadovskii's principle, Krasnoselskii's theorem, and the nonlinear alternative Leray-Schauder technique. Finally, numerical examples are provided for each problem discussed.

Keywords : Fractional differential equation, Integral boundary condition, Existence, Uniqueness, Fixed point theorems, .

Résumé

Dans cette note, nous définissons certains types de dérivées fractionnaires, telles que celles de Riemann-Liouville, Caputo et Hadamard, tout en présentant les différentes propriétés associées à ces dérivées fractionnaires. Ensuite, nous étudions l'existence et l'unicité des solutions pour certains problèmes représentés par des équations différentielles fractionnaires d'ordres multiples, tels que les problèmes aux limites fractionnaires (**FBVP**).

Ces problèmes impliquent des dérivées fractionnaires comme celles de **Caputo-Hadamard**, ainsi que des **conditions aux limites non locales** dans des **espaces de Banach**. Les résultats obtenus dans cette étude reposent sur des théorèmes de point fixe : le **principe de contraction de Banach**, le **principe de Sadovskii**, le **théorème de Krasnoselskii**, et la **technique non linéaire de Leray-Schauder**. Enfin, des exemples numériques sont fournis pour chaque problème abordé.

Mots clés : Equation différentielle fractionnaire, Condition aux limites intégrale, Existence, Unicité, Théorèmes de point fixe, .

ملخص.

في هذه المذكرة قمنا بتعريف بعض انواع المشتقات الكسرية مثل ريمان و كابوتو و هدامار مع تقديم مختلف الخواص المتعلقة بهذه المشتقات الكسرية ،ثم قمنا بدراسة وجود وتفرد حلول بعض المسائل التي تمثلها المعادلات التفاضلية الكسرية ذات الرتب المتعددة مثل المسائل الحدية الكسرية (FBVP) التي تتضمن مشتقات كسرية مثل و Caputo-Hadamard مع شروط حدية غير محلية في فضاءات بناخ تستند النتائج التي تم الحصول عليها في هذه الدراسة على نظريات النقطة الثابتة: مبدأ انكماش Banach ومبدأ Sadowskii و Krasnoselski وتقنية الاستبدال غير الخطي Leray-Schauder وأخيرًا، يتم إعطاء أمثلة عددية لكل مشكلة يتم تناولها.

الكلمات المفتاحية: المعادلة التفاضلية الكسرية، معادلة
شرط الحد التكاملي، الوجود، التفرد، نظريات النقطة الثابتة،.