**République Algérienne Démocratique et Populaire**
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

---

Université de Ahmed Zabana de Relizane

**Faculté Mathématiques et informatiques**
**Département de Mathématiques**

# MEMOIRE

**En vue de l'obtention du diplôme de MASTER en:**
**Géométrie différentielle**

**Intitulé**

# Dynamics of wave equations with critical nonlinearity

**Présenté par :**

**Mr** : Kouider Daouadji Djamel

**Devant les membres de jury :**

| | | |
|---|---|---|
| **Présidente** : Mme Boussekkine Naima | Maître de conférences (A) | ( U. **Relizane**) |
| **Encadrant** : Mme Remil Melouka | Maître de conférences (A) | (U. **Relizane**) |
| **Examinateur** : Mlle Mahdi fatima zohra | Maître de conférences (B) | (U. **Relizane**) |

**Année universitaire : 2024/2025**

# Abstract

This work investigates the existence, uniqueness, and long-time behavior of solutions to nonlinear partial differential equation using functional and variational methods. We begin with a presentation of the functional spaces, as well as key inequalities fundamental to the analysis of PDEs. The Faedo-Galerkin method is introduced as a primary tool for constructing approximate solutions.

We then focus on evolution equation with critical nonlinearity arising in mathematical physics, analyzing the existence and blow-up of solutions. The existence theory is developed both locally and globally, supported by appropriate energy functionals and a priori estimates. We also prove uniqueness under general assumptions.

The final part of the work is dedicated to the long-time dynamics of dissipative systems, we study the generation of dynamical systems in appropriate phase spaces, analyze the asymptotic regularity of solutions, and establish the existence of global and exponential attractors. This provides a comprehensive picture of the qualitative behavior of solutions as time tends to infinity, including stability. Our results contribute to the understanding of the global structure of the solution set and the mechanisms that govern dissipation and regularity in nonlinear PDEs.

# Acknowledgement

First of all, I would like thank our God for his conciliation to me to achieve this point otherwise.

Foremost, I would like to express my sincere gratitude to my advisor Professor **REMIL MELOUKA** for the continuous support of my study and research, for her patience, motivation, and immense knowledge. Her guidance helped me in all the time of research and writing of this work.

My deepest gratitude goes to Professor **BOUSSEKKINE NAIMA** who accepted to chair my committee.

I would like to thank Professor **MAHDI FATIMA ZOHRA**, and it is a great honour for me she is part of the jury.

Finally, my sincere thanks to all the professors of the department of mathematics at the university of Relizane.

Thank you to all those who have contributed, before or until now, to the success of this work.

# Dedicate

To my beloved parent, whose unwavering support, endless sacrifices, and unconditional love have been the foundation of all my achievements.

To my dear family,
for always believing in me and standing by my side through every challenge.

To my professors and mentors,
who guided me with wisdom and inspired me with their dedication to knowledge.

To my friends,
whose encouragement and companionship made this journey more meaningful and memorable.

And to everyone who has been a part of this path,
I dedicate this work as a token of my deep appreciation and gratitude.

Finally , I ask my god almighty that this work be in the balance of my mother good deeds , my god have mercy on her .

*Kouider Daouadji Djamel*

# Contents

## 0.1 Introduction

Wave equations are essential in mathematical physics, allowing for the modeling of various phenomena such as acoustic waves, electromagnetic waves, and mechanical vibrations.Nonlinear effects can greatly affect the propagation of waves in many applications, resulting in significant changes in their behavior.To control wave propagation in fields like nonlinear optics, material science, and general relativity, it is crucial to understand these nonlinear dynamics.

This work concentrates on the study of the dynamics of wave equation with critical nonlinearity, where the intaction between dispersion, damping, and nonlinear term product complex phenomena such as global existence, finite-time blow-up, and long-time asymptotic action.
The equation that being studied is given by the following formula:

$$u_{tt}(x,t) - \Delta u + |u_t(x,t)|^{q-1}u_t(x,t) + f(u) = 0, \quad x \in \Omega, \quad q > 1, t \geq 0 \tag{1.1}$$

with boundary and initial conditions:

$$u|_{\partial\Omega} = 0, \quad u(0) = u_0, \quad u_t(0) = u_1. \tag{1.2}$$

Were, $\Omega$ is bounded domain in $\mathbb{R}^n$, $\quad n \geq 1$ with $|\Omega| < \infty$, $u(x,t)$, $|u_t|^{q-1}u_t$ is a nonlinear damping term, $f(u)$ is the critical nonlinear term. The exponent $q > 1$ is controled the damping effect.

Wave equations with critical nonlinearity are a rich mathematical area that has open problems regarding global existence, compactness, and asymptotic behavior. This work intends to improve understanding by combining functional analysis and variational methods Non-linear wave dynamics contribute to both theoretical and applied aspects .

Numerous mathematical difficulties arise when studying this wave equation One of the primary The challenge is to determine the conditions for a global existence and an explosion in time. Some initial data configurations may lead to singularities, requiring accurate energy estimates to understand the behavior of the solution.

Besides, critical non-linearities cause the failure of standard Sobolev integrations, this leads to a loss of compactness. to control them, advanced techniques are needed. Another key issue is to understand the long-term dynamics of solutions, whether they are stabilized, dispersed or oscillating. Energy methods play a decisive role in addressing these concerns, helping to establish inequalities that control the solution's behavior and ensure good results.

The purpose of our work is the study of wave equations with critical non-linearity by addressing navious aspects. First, we to set the existence and uniqueness of solutions by building appropriate functional spaces and variational frameworks. Then, we study the mechanisms of swelling by identifying critical thresholds for energy concentration. In addition, asymptotic analysis using some estimates will be used to study diffusion and stability teatures.

## Structure of the Thesis

Our work is organized as follows:

**Chapter 1:** Preliminaries.

This chapter introduces the functional spaces and mathematical tools required throughout the work. We begin with a detailed presentation of functional spaces, including Banach and Hilbert spaces, $L^p$ and Sobolev spaces, which provide the appropriate framework for studying solutions of PDEs. Fundamental inequalities such as Hölder, Sobolev, and Poincaré inequalities are reviewed. We also introduce the Faedo-Galerkin method, a technique for proving existence results. Finally, we provide a brief overview of dynamical systems concepts such as semigroups and attractors, which are essential for the analysis of long-time behavior in later chapters.

**Chapter 2:** Existence and uniqueness of solution.

In this chapter we will define energy and prove that the energy function decrease over time, in addition we prove both local and global existence theorems under general hypotheses. At the end of this chapter we will prove the uniqueness of the solution.

**Chapter 3:** Long-time dynamics.

In the final chapter, we analyze the asymptotic behavior of solutions as time tends to infinity. Using the semigroup framework, we demonstrate the generation of a dynamical system associated with the PDE. We then study the asymptotic regularity of trajectories and the existence of absorbing sets, which characterize the dissipative nature of the system. Finally, we construct a uniform exponential attractor, providing a compact, finite-dimensional description of the long-time dynamics. These results give insight into the global stability for the class of equations considered .

# Chapter 1

# Preliminaries .

## 1.1 Functional Spaces

In this chapter, we will introduce and state without proofs some important materials needed in the proof of our results,

### 1.1.1 Banach Spaces

**Définition 1.1.1.** *A Banach space is a normed vector space* $(E, \|\cdot\|)$ *that is complete, meaning that every Cauchy sequence in* $E$ *converges to a limit in* $E$.

*In other words, if* $(x_n) \subset E$ *satisfies:*

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ such that } \|x_n - x_m\| < \varepsilon, \quad \forall n, m \geq N,$$

*then there exists* $x \in E$ *such that:*

$$\|x_n - x\| \to 0 \quad \text{as } n \to +\infty.$$

**Définition 1.1.2.** *Distance in Banach Space*

*The* ***Hausdorff semi-distance*** *in Banach space* $\mathbb{E}$ *is defined as:*

$$dist_{\mathbb{E}}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_{\mathbb{E}}$$

*for any two subsets* $A, B \subseteq \mathbb{E}$, *where* $\|\cdot\|_{\mathbb{E}}$ *is the norm on* $\mathbb{E}$.

**Théorème 1.1.3.** *([35])* ***Hahn-Banach Theorem.***
*Let* $p : X \to \mathbb{R}$ *be a sublinear function, and let* $f$ *be a linear functional defined on a subspace* $Y \subset X$ *such that:*

$$f(y) \leq p(y), \quad \forall y \in Y.$$

*Then there exists a linear extension* $F : X \to \mathbb{R}$ *of* $f$ *such that:*

$$F(x) \leq p(x), \quad \forall x \in X.$$

**Théorème 1.1.4.** *([36])* ***Uniform Boundedness Principle (Banach-Steinhaus).***
*Let* $\{T_\alpha\}, \forall \alpha \in \mathbb{N}$ *be a family of bounded linear operators from a Banach space* $X$ *to a normed space* $Y$. *If for every* $x \in X$, *the set* $\{\|T_\alpha x\|\}$ *is bounded, then:*

$$\sup_{\alpha} \|T_\alpha\| < +\infty.$$

**Théorème 1.1.5.** *([36])* ***Open Mapping Theorem.***
*If $X$ and $Y$ are Banach spaces, and $T : X \to Y$ is a bounded linear operator that is surjective, then $T$ is an open map (i.e, it maps open sets in $X$ to open sets in $Y$).*

**Théorème 1.1.6.** *([36])(Banach-Alaoglu)*

*Let $X$ be a normed vector space, and let $X^*$ denote its dual space (the space of all continuous linear functionals on $X$).*

*Then the closed unit ball of $X^*$, defined by*

$$B_{X^*} = \{f \in X^* : \|f\| \leq 1\},$$

*is compact in the weak* topology.*

### 1.1.2 Hilbert spaces

**Définition 1.1.7.** *A Hilbert space $H$, is a vectorial space supplied with inner product $\langle u, v\rangle$, such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let $H$ complete.*

**Théorème 1.1.8.** *([39]) Let $(u_n)_{n\in N}$ is a bounded sequence in the Hilbert space $H$, it posses a subsequence which converges in the weak topology of $H$.*

**Théorème 1.1.9.** *([39]) In the Hilbert space, all sequence which converges in the weak topology is bounded.*

**Théorème 1.1.10.** *([38]) Let $(u_n)_{n\in N}$ be a sequence which converges to $u$, in the weak topology and $(v_n)_{n\in N}$ is an other sequence which converge weakly to $v$, then*

$$\lim_{n\to\infty} \langle v_n, u_n\rangle = \langle v, u\rangle$$

### The $L^p(\Omega)$ spaces

**Définition 1.1.11.** *Let $1 \leq p < +\infty$ and let $\Omega$ be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p dx < +\infty \right\}.$$

*If $p = +\infty$, we have:*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \text{ is measurable and there exists a constant } C \text{ such that } |f(x)| \leq C \text{ a.e in } \Omega\}.$$

**Notation 1.1.12.** *We denote:*

$$\|f\|_p = \Big[\int_\Omega |f(x)|^p dx\Big]^{\frac{1}{p}}.$$

$$\|f\|_\infty = \inf\{C,\ |f(x)| \leq C \text{ a.e in } \Omega\}.$$

**Notation 1.1.13.** *For $p \in \mathbb{R}$ and $1 \leq p \leq +\infty$, we denote by $q$ the conjugate of $p$ i.e. $\frac{1}{p} + \frac{1}{q} = 1$.*

**Théorème 1.1.14.** *([38]) $L^p$ is a vectorial space, and $\|.\|_p$ is a norm for all $1 \leq p \leq +\infty$,*

**Théorème 1.1.15.** *([38])(Fischer-Riesz) $L^p$ is a Banach space for all $1 \leq p \leq +\infty$.*

**Remarque 1.1.16.** *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx,$$

*is a Hilbert space.*

### 1.1.3 Sobolev spaces

The theory of Sobolev spaces has been developed by generalizing the notion of classical derivatives and introducing the idea of weak or generalized derivatives.

**Definition and basic properties**

Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $n \geq 1$. Let $\alpha$ a multi-index where $\alpha = (\alpha_1, ..., \alpha_d)$ with $\alpha_i \geq 0$, for any $i \in 1, ..., d, |\alpha| = \sum_{i=1}^{d} \alpha_i$ and $D^\alpha = D_1^{\alpha_1} ... D_d^{\alpha_d}$ with $D_i = \frac{\partial}{\partial x_i}$.

**Proposition 1.1.17.** *([38]) Let $\Omega$ be an open domain in $\mathbb{R}^n$ and $n \geq 1$. Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx, \text{ for all } \varphi \in D(\Omega),$$

*where $1 \leq p \leq +\infty$ and it's well-known that $f$ is unique.*

**Définition 1.1.18.** *Let $1 \leq p \leq +\infty$, and $k \in \mathbb{N}$.*
*The Sobolev space $W^{k,p}(\Omega)$ is the space of functions $f \in L^p(\Omega)$. we defined:*

$$W^{k,p}(\Omega) = \left\{ f \in L^p(\Omega), D^\alpha f \in L^p(\Omega). \ \forall \alpha \in \mathbb{N}^d, |\alpha| \leq k \right\}.$$

*For $k = 0$, we set $W^{0,p}(\Omega) = L^p(\Omega)$.*
*The space $W^{k,p}(\Omega)$ becomes a Banach space with the norm:*

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}, \text{ for } 1 \leq p < +\infty. \\ \\ \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}, \text{ for } p = +\infty \end{cases}$$

*$W^{k,p}(\Omega)$ is a reflexive space for $1 < p < +\infty$, and a separable space for $1 \leq p < +\infty$.*

**Définition 1.1.19.** *Let $1 \leq p \leq +\infty$ and $k \in \mathbb{N}$. Then, the Sobolev space $W_0^{k,p}(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ in the norm of the space $W^{k,p}(\Omega)$. It follows from the definition above that the space $W_0^{k,p}(\Omega)$ is a Banach space with the norm $\|.\|_{W^{k,p}(\Omega)}$.*

*We write $H_0^k(\Omega) = W_0^{2,p}(\Omega)$.*

**Théorème 1.1.20.** *([38]) Let $\Omega$ be an open bounded domain of $\mathbb{R}^n$ and $n \geq 1$ with a Lipschitz boundary. For nonnegative integers $k, l$ such that $0 \leq l \leq k$, we have the continuous embeddings $W^{k,p}(\Omega) \subset W^{l,p}(\Omega)$ for all $1 \leq p \leq +\infty$.*
*Moreover, for $k \geq 0$, we have $W^{k,r}(\Omega) \subset W^{k,p}(\Omega)$ for all $1 \leq p \leq r \leq +\infty$.*

**Définition 1.1.21.** *If $1 \leq p < n$, the Sobolev conjugate of $p$ is defined as*

$$p^* = \frac{np}{n-p}.$$

*Equivalently, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Also, $p^* > p$.*

**Lemme 1.1.22.** *([38])(Sobolev-Poincaré's inequality) Let* $\Omega$ *be a bounded open subset of* $\mathbb{R}^n$ *and* $n \geq 1$*, then there is a constant* $C(p, \Omega)$ *(depending on* $p$ *and* $\Omega$*) such that*

$$\text{If} \quad 2 \leq p \leq \frac{2n}{n-2}, n \geq 3 \quad \text{and} \quad q \geq 2, n = 1, 2,$$

*then*

$$\|u\|_{L^p(\Omega)} \leq C(p, \Omega) \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

**Proposition 1.1.23.** *([39])( Green's formula) For all* $u \in H^2(\Omega)$*,* $\upsilon \in H^1(\Omega)$ *we have*

$$-\int_\Omega \Delta u \upsilon dx = \int_\Omega \nabla u \nabla \upsilon dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \upsilon d\sigma,$$

*where* $\frac{\partial u}{\partial \eta}$ *is a normal derivation of* $u$ *at* $\Gamma$*.*

## 1.2 Some integral and algebraic inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here.

**Théorème 1.2.1.** *([3]) (Hölder's inequality) Let* $1 \leq p \leq +\infty$*. Assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$*, then* $fg \in L^1(\Omega)$ *and*

$$\int_\Omega |fg| dx \leq \|f\|_p \|g\|_q.$$

**Proposition 1.2.2.** *([40]) If* $\mu(\Omega) < \infty$*,* $1 < p < q < +\infty$*, then* $L^q \hookrightarrow L^p$ *and, if* $u \in L^q(\Omega)$

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^q}.$$

*if* $u \in L^\infty(\Omega)$ *then* $u \in L^p(\Omega)$ *and*

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p}} \|u\|_{L^\infty}.$$

**Lemme 1.2.3.** *([39])The Cauchy-Schwartz's inequality) Every inner product satisfies the Cauchy-Schwartz's inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\|.$$

*The equality sign holds if and only if* $x_1$ *and* $x_2$ *are dependent.*

**Lemme 1.2.4.** *([39])(Young's inequalities) For all* $a, b \in \mathbb{R}^+$*, we have*

$$ab \leq \alpha a^2 + \frac{1}{4\alpha} b^2,$$

*where* $\alpha$ *is any positive constant.*

**Lemme 1.2.5.** *([39]) For* $a, b \geq 0$*, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

*where,* $\frac{1}{p} + \frac{1}{q} = 1$*.*

**Théorème 1.2.6.** *([42]) Dunford-Pettis Theorem in Sobolev Spaces*

*If $u \subset W^{1,p}(\Omega)$ is* ***bounded*** *and its gradients $\nabla u$ is* ***integrable****, then $u$ is relatively weakly compact in $W^{1,p}(\Omega)$.*

**Lemme 1.2.7.** *([43])( Zorn's Lemma)*

*Let $(S, \leq)$ be a* ***partially ordered set*** *such that every totally ordered subset has an* ***upper bound*** *in $S$.*

*Then, $S$ contains at least one* ***maximal element****, that is, an element $m \in S$ such that there is no $x \in S$ with $m < x$.*

**Lemme 1.2.8.** *([44]) (Aubin–Lions Lemma) Let $X_0 \subset X \subset X_1$ be three Banach spaces such that:*

- *the embedding $X_0 \hookrightarrow X$ is compact,*
- *the embedding $X \hookrightarrow X_1$ is continuous.*

*Let $1 \leq p, q \leq +\infty$. Then the set*

$$\left\{ u \in L^p(0,T;X_0) \,\middle|\, \frac{\partial u}{\partial t} \in L^q(0,T;X_1) \right\},$$

*is relatively compact in $L^p(0,T;X)$, provided that one of the following holds:*

- *$p < +\infty$,*
- *$p = +\infty$ and $q > 1$.*

## 1.3 Faedo-Galerkin Method for PDEs

The Faedo-Galerkin method is an approximation technique used to prove the existence of weak solutions to partial differential equations (PDEs).

Consider the heat equation:

$$u_t - \Delta u = f \quad \text{in } \Omega \times (0,T),$$

with initial condition $u(x,0) = u_0(x)$ and homogeneous Dirichlet boundary conditions.

The weak formulation seeks $u \in H^1(\Omega)$ such that:

$$\int_\Omega u_t v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx, \quad \forall v \in H_0^1(\Omega).$$

Consider a finite-dimensional subspace $V_n = \operatorname{span}\{\phi_1, \dots, \phi_n\} \subset H_0^1(\Omega)$.

Construct an approximate solution of the form:

$$u_n(x,t) = \sum_{k=1}^n c_k^n(t) \phi_k(x).$$

Substituting into the weak formulation, we obtain:

$$\sum_{k=1}^n \left( \int_\Omega \phi_j \phi_k \, dx \right) \frac{dc_k^n}{dt} + \sum_{k=1}^n \left( \int_\Omega \nabla \phi_j \cdot \nabla \phi_k \, dx \right) c_k^n = \int_\Omega f \phi_j \, dx.$$

This gives a system of ODEs .

Using standard ODE existence theorems, the system has a solution.

Energy estimates give uniform bounds on $u_n$, ensuring weak compactness.

Using compactness results, we extract a weakly convergent subsequence:

$$u_n \rightharpoonup u \quad \text{in } L^2(0,T; H_0^1(\Omega)).$$

Thus, $u$ is a weak solution.

**Théorème 1.3.1.** *([45])(Gronwall Inequality)*
*Let $u(t)$, $\alpha(t)$, and $\beta(t)$ be continuous real-valued functions on an interval $[0,T]$, and suppose:*

$$u(t) \leq \alpha(t) + \int_0^t \beta(s) u(s) \, ds, \quad \text{for all } t \in [0,T].$$

*Then:*

$$u(t) \leq \alpha(t) + \int_0^t \alpha(s) \beta(s) \exp\left( \int_s^t \beta(r) \, dr \right) ds.$$

*In particular, if $\alpha(t) = C$ is constant we have:*

$$u(t) \leq C \exp\left( \int_0^t \beta(s) \, ds \right).$$

## 1.4 Attractor and semigroup

**Définition 1.4.1** (Dynamical System)**.** *Let $\mathcal{H}$ be a Banach (or metric) space. A* ***dynamical system*** *is a pair $(S(t), \mathcal{H})$, where $\{S(t)\}_{t\geq 0}$ is a family of mappings*

$$S(t) : \mathcal{H} \to \mathcal{H}, \quad t \geq 0,$$

*satisfying:*

- ***(i)*** $S(0) = \mathrm{Id}$ *(identity on $\mathcal{H}$).*
- ***(ii)*** $S(t+s) = S(t) \circ S(s)$ *for all $t, s \geq 0$ (semigroup property).*
- ***(iii)*** *The map $t \mapsto S(t)x$ is continuous from $[0, +\infty)$ into $\mathcal{H}$ for each fixed $x \in \mathcal{H}$.*

**Définition 1.4.2.** *(Semigroup of Operators)*
*Let $\mathcal{H}$ be a Banach space and let $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ be a closed linear operator. We assume that $A$ generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$, that is:*

- $S(0) = I$ *(the identity),*
- $S(t+s) = S(t)S(s)$ *for all $t, s \geq 0$,*
- *The map $t \mapsto S(t)x$ is continuous on $[0, +\infty)$ for every $x \in \mathcal{H}$.*

*The semigroup $\{S(t)\}_{t\geq 0}$ represents the solution of the abstract Cauchy problem:*

$$\begin{cases} \dfrac{du}{dt} = Au, & t > 0, \\ u(0) = u_0 \in \mathcal{H}. \end{cases}$$

**Définition 1.4.3.** *(Absorbing Set)*
*Let $\mathcal{H}$ be a Banach space, and let $\{S(t)\}_{t\geq 0}$ be a semigroup acting on $\mathcal{H}$, an* ***absorbing set*** *$B_0 \subset \mathcal{H}$ is a bounded set such that:*
*For every bounded subset $B \subset \mathcal{H}$, there exists a time $T_B > 0$ such that*

$$S(t)B \subset B_0 \quad \textit{for all } t \geq T_B.$$

## Semigroup Stability

Several types of stability can be distinguished for the semigroup $\{S(t)\}_{t\geq 0}$:

### • Strong Stability

The semigroup is said to be **strongly stable** if

$$\lim_{t\to\infty} \|S(t)x\| = 0, \quad \text{for all } x \in \mathcal{H}.$$

### • Exponential Stability

The semigroup is **exponentially stable** if there exist constants $M \geq 1$ and $\omega > 0$ such that

$$\|S(t)\| \leq Me^{-\omega t}, \quad \text{for all } t \geq 0.$$

This implies that the solutions decay exponentially in time, uniformly.

**Théorème 1.4.4.** *([46]) (Gearhart-Prüss Theorem)*
*Suppose that $H$ is a Hilbert space and that $A$ generates a semigroup of contractions. Then, the semigroup $\{S(t)\}$ is exponentially stable if and only if:*

- $\sigma(A) \subset \{\lambda \in \mathbb{C} \mid \Re\lambda < 0\}$,
- *and the following resolvent condition is satisfied:*

$$\sup_{\Re\lambda > 0} \|(\lambda I - A)^{-1}\| < \infty.$$

**Définition 1.4.5.** *(Global Attractor) Let $(S(t))_{t\geq 0}$ be a semigroup of operators defining a dynamical system on a Banach space $\mathcal{H}$. A set $\mathcal{A} \subset \mathcal{H}$ is called a **global attractor** if it satisfies the following properties:*

- ***Invariance**:*

$$S(t)\mathcal{A} = \mathcal{A}, \quad \forall t \geq 0.$$

- ***Attracting property**:*

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t)B, \mathcal{A}) = 0, \quad \textit{for every bounded set } B \subset \mathcal{H}.$$

- ***Compactness**: $\mathcal{A}$ is compact in $\mathcal{H}$.*

*The set $\mathcal{A}$ characterizes the long-time behavior of the dynamical system and attracts all bounded subsets of the phase space.*

**Définition 1.4.6** (Time-dependent Exponential Attractor)**.** *We call the family $\mathcal{M} = \{\mathcal{M}(s) \mid s \in \mathbb{R}\}$ a time-dependent exponential attractor for the semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{B}_S$ if:*

1. *There exists $0 < \varpi < +\infty$ such that $\mathcal{M}(s) = \mathcal{M}(\varpi + s), \quad \forall s \in \mathbb{R}$;*
2. *The subset $\mathcal{M}(s) \subset \mathcal{B}_S$ is non-empty and compact, and the fractal dimension of the set $\mathcal{M}(s)$ is uniformly bounded, $\quad \forall s \in \mathbb{R}$;*
3. *The family is positive semi-variant, that is,*

$$S(t)\mathcal{M}(s) \subset \mathcal{M}(t+s), \quad \forall t \geq 0, \ \forall s \in \mathbb{R};$$

4. *There exist two positive constants $\alpha$ and $\beta$ such that*

$$\sup_{s\in[0,\varpi]} dist_{\mathcal{E}}(S(t)\mathcal{B}_S, \mathcal{M}(s)) \leq \alpha e^{-\alpha t}, \quad \forall t \geq 0.$$

**Définition 1.4.7.** *Exponential Attraction Inequality*
*For the dynamical system $S(t)$ with absorbing set $\mathcal{B}$ and global attractor $\mathcal{M}$, we have:*

$$dist_{\mathbb{E}}\left(S(t)\mathcal{B}, \mathcal{M} \times \mathbb{R}^n\right) \leq \mathcal{Q}(\|\mathcal{B}\|_{\mathbb{E}})e^{-\alpha t}, \quad \text{for all } t \geq 0,$$

*where:*

- $\mathcal{Q} : [0, \infty) \to [0, \infty)$ *is a monotone increasing function.*
- $\alpha > 0$ *is the exponential decay rate.*
- $\|\mathcal{B}\|_{\mathbb{E}} = \sup_{x\in\mathcal{B}} \|x\|_{\mathbb{E}}$ *is the radius of* $\mathcal{B}$.

**Théorème 1.4.8.** *([37]) Let $\mathcal{B}_S$ be a bounded closed subset of Banach space $E$, and $(S(t), \mathcal{B}_S, E)$ be an autonomous dynamical system. And assume that*

1. *There exist constants $T > 0$ and $L_T > 0$ such that*

$$\begin{aligned} S(t)\mathcal{B}_S \subset \mathcal{B}_S, &\quad \forall t \geq T, \\ \|S(t)x - S(t)y\|_E \leq L_T\|x - y\|_E, &\quad \forall x, y \in \mathcal{B}_S, \end{aligned}$$

2. *There exist a positive $t^*$, a compact seminorm $n_Z(\cdot)$ on $Z$ and a mapping $C(t^*) : \mathcal{B}_S \to Z$ such that*

$$\begin{aligned} \|C(t^*)x - C(t^*)y\|_Z \leq L_C\|x - y\|_E, &\quad \forall x, y \in \mathcal{B}_S; \\ \|S(t^*)x - S(t^*)y\|_E \leq \theta\|x - y\|_E + n_Z(C(t^*)x - C(t^*)y), &\quad \forall x, y \in \mathcal{B}_S, \end{aligned}$$

*where $0 < \theta < \frac{1}{2}$, $L_C > 0$ are constants.*
*Then for all $\kappa \in (0, \frac{1}{2} - \theta)$, the dynamical system $(S(t), \mathcal{B}_S, E)$ possesses a time-dependent exponential attractor $\mathcal{M} = \{\mathcal{M}^\kappa(t) \mid t \in \mathbb{R}\}$.*
*Moreover,*

$$\dim_F^E(\mathcal{M}^\kappa(t)) \leq \log_{\frac{1}{2(\kappa+\theta)}}\left(N^{n_Z}_{\frac{\kappa}{L_C}}\left(B_1^Z(0)\right)\right), \quad \text{for all } t \in \mathbb{R},$$

*where $B_r^Z(a)$ denotes the ball of radius $r > 0$ and center $a \in Z$ in the metric space $Z$.*

*and $N_\epsilon^{n_Z}(A)$ denotes the minimal number of $\epsilon$-balls with centers in $A$ needed to cover the subset $A \subset Z$ with seminorm $n_Z$.*

# Chapter 2

# Existence and uniqueness of solution.

## 2.1 Existence of Solutions

We consider the following wave equation with critical non linearity

$$u_{tt}(x,t) - \Delta u + |u_t(x,t)|^{q-1}u_t(x,t) + f(u) = 0, \quad x \in \Omega, \quad q > 1, t \geq 0, \tag{1.1}$$

with boundary and initial conditions:

$$u|_{\partial\Omega} = 0, \quad u(0) = u_0, \quad u_t(0) = u_1. \tag{1.2}$$

The goal is to show that there exists a unique solution in a suitable functional space.

### 2.1.1 Energy Functional

For the given equation:

$$u_{tt} - \Delta u + |u_t|^{q-1}u_t + f(u) = 0,$$

the **energy functional** $E(t)$ is defined by the formula:

$$E(t) = \frac{1}{2}\int_\Omega \left(u_t^2 + |\nabla u|^2\right) dx + \int_\Omega F(u)\, dx,$$

where

$$F(u) = \int_0^u f(s)\, ds.$$

Consider the time derivative of the energy functional $E(t)$:

$$\frac{dE}{dt} = \int_\Omega (u_t u_{tt} + \nabla u \cdot \nabla u_t)\, dx + \int_\Omega f(u) u_t\, dx.$$

Substitute $u_{tt} = \Delta u - |u_t|^{q-1}u_t - f(u)$ in $\frac{dE}{dt}$ we get :

$$\frac{dE}{dt} = \int_\Omega (u_t \Delta u + \nabla u \cdot \nabla u_t)\, dx - \int_\Omega |u_t|^{q+1} dx.$$

Using green formula , we get:

$$\frac{dE}{dt} = -\int_\Omega |u_t|^{q+1} dx.$$

Then, $\frac{dE}{dt} \leq 0$, and $E(t)$ is non-increasing, then the energy $E(t)$ is bounded above by its initial value:

$$E(t) \leq E(0) \quad \text{for all } t \geq 0.$$

### 2.1.2 Local Existence Theorem

**Théorème 2.1.1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that:*
*The initial data satisfy $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, and $q > 1$.*
*The nonlinear term $f(u)$ is locally Lipschitz continuous.*

*Then, there exists a time $T > 0$ and a unique solution $u(t)$ to the problem (1.1)-(1.2) such that:*

$$u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)), \quad u_t \in C([0,T]; L^2(\Omega)).$$

*Proof.* Let $\{w_j\}_{j=1}^\infty$ be an orthonormal basis of $H^2(\Omega) \cap H_0^1(\Omega)$ consisting of eigenfunctions of the Laplacian $-\Delta$ with Dirichlet boundary conditions. Define the finite-dimensional subspace:

$$V_k = \text{span}\{w_1, w_2, \ldots, w_k\}.$$

Construct the approximate solution $u_k(t)$ in the form:

$$u_k(t) = \sum_{j=1}^k g_{jk}(t) w_j,$$

where $g_{jk}(t)$ are time-dependent coefficients. The initial conditions for $u_k(t)$ are:

$$u_k(0) = \sum_{j=1}^k \langle u_0, w_j \rangle w_j, \quad \partial_t u_k(0) = \sum_{j=1}^k \langle u_1, w_j \rangle w_j.$$

Product the equation equation (1.1) in $w_j$ and integrate over $\Omega$, where the coefficients $g_{jk}(t)$, $j = 1, 2, \ldots, k$, are determined by solving the following system of ordinary differential equations (ODEs) :

$$\int_\Omega \left( u_{k,tt} w_j - \Delta u_k w_j + |u_{k,t}|^{q-1} u_{k,t} w_j + f(u_k) w_j \right) dx = 0,$$

Define the energy functional for $u_k(t)$:

$$E_k(t) = \frac{1}{2}\|u_{k,t}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_k\|_{L^2}^2 + \int_\Omega F(u_k) dx,$$

Where $F(u) = \int_0^u f(s) ds$.
Differentiate $E_k(t)$ with respect to $t$:

$$\frac{d}{dt} E_k(t) = \int_\Omega \left( u_{k,t} u_{k,tt} + \nabla u_k \cdot \nabla u_{k,t} + f(u_k) u_{k,t} \right) dx.$$

Using the same procedure as the proof of decreasing energy we find:

$$\frac{d}{dt}E_k(t) \leq -C\|u_{k,t}\|_{L^{q+1}}^{q+1}.$$

This implies that $E_k(t)$ is bounded uniformly in $k$, and:

$$\|u_k\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \quad \|u_{k,t}\|_{L^\infty(0,T;L^2(\Omega))} \leq C.$$

By the Banach-Alaoglu theorem, extract a subsequence $\{u_k\}$ such that:

$$u_k \rightharpoonup u \quad \text{weakly-* in } L^\infty(0,T;H^1(\Omega)),$$

$$u_{k,t} \rightharpoonup u_t \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)).$$

Using Aubin-Lions lemma, we have strong convergence:

$$u_k \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Pass to the limit in the weak formulation to show that $u$ satisfies:

$$u_{tt} - \Delta u + |u_t|^{q-1}u_t + f(u) = 0.$$

□

### 2.1.3 Global Existence Theorem

**Théorème 2.1.2.** *Under the same assumptions as in Theorem(2.1.1), and additionally:*

*The nonlinear term $f(u)$ satisfies the condition:*

$$|f(u)| \leq C(1+|u|^p) \quad \textit{for some } p \geq 1,$$

*and the potential energy $F(u) = \int_0^u f(s)\,ds$ satisfies:*

$$F(u) \geq -C_1|u|^2 - C_2 \quad \textit{for some constants } C_1, C_2 \geq 0.$$

*Then, the solution $u(t)$ exists globally in time, i.e., for all $t \geq 0$, and satisfies:*

$$u \in L^\infty(0,\infty;H^2(\Omega)\cap H_0^1(\Omega)), \quad u_t \in L^\infty(0,\infty;L^2(\Omega)) \cap L^{q+1}(\Omega\times(0,\infty)).$$

*Proof.* We aim to extend the local solution globally in time. The proof relies on uniform a priori estimates and a continuation argument.

Recall the energy functional defined for the Galerkin approximate solution $u_k(t)$:

$$E_k(t) = \frac{1}{2}\|u_{k,t}(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla u_k(t)\|_{L^2(\Omega)}^2 + \int_\Omega F(u_k(t))\,dx,$$

Where $F(u) = \int_0^u f(s)\,ds$.

We have that $E_k(t)$ is non-increasing in time and

$$E_k(t) \leq E_k(0) \quad \text{for all } t \in [0,T].$$

By the growth assumption on $F$, we obtain

$$\int_\Omega F(u_k(t))\,dx \geq -C_1\|u_k(t)\|^2_{L^2(\Omega)} - C_2|\Omega|.$$

Combining this with the energy estimate, we get

$$\frac{1}{2}\|u_{k,t}(t)\|^2_{L^2} + \frac{1}{2}\|\nabla u_k(t)\|^2_{L^2} \leq E_k(0) + C_1\|u_k(t)\|^2_{L^2} + C_2|\Omega|.$$

Using Poincaré's inequality, we have:

$$\|\nabla u_k(t)\|^2_{L^2} \leq C(E_k(0)+1).$$

Similarly, $\|u_{k,t}(t)\|^2_{L^2} \leq C(E_k(0)+1)$. Thus, the solution and its time derivative are bounded uniformly in $k$ and $t \in [0,T]$, and the dissipation term satisfies:

$$\int_0^T \|u_{k,t}(s)\|^{q+1}_{L^{q+1}(\Omega)}\,ds \leq E_k(0).$$

As in the local existence proof, using the uniform estimates, we extract weak-* convergent subsequences:

$$u_k \rightharpoonup u \quad \text{weakly-* in } L^\infty(0,T;H_0^1(\Omega)), \quad u_{k,t} \rightharpoonup u_t \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)),$$

$$u_{k,t} \rightharpoonup u_t \quad \text{weakly in } L^{q+1}(0,T;L^{q+1}(\Omega)).$$

Moreover, by the Aubin-Lions compactness lemma, we obtain strong convergence:

$$u_k \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Up to a subsequence. This allows us to pass to the limit in the nonlinear terms $f(u_k) \to f(u)$, $|u_{k,t}|^{q-1}u_{k,t} \rightharpoonup |u_t|^{q-1}u_t$, using weak-strong convergence arguments.

The local solution $u(t)$ exists up to some maximal time $T^* > 0$. If $T^* < +\infty$, then the uniform estimates above imply that

$$\sup_{t\in[0,T^*)} \|u(t)\|^2_{H^1(\Omega)} + \|u_t(t)\|^2_{L^2(\Omega)} < +\infty.$$

Thus, there is no blow-up in finite time, and the solution can be continued beyond $T^*$, contradicting maximality. Hence, $T^* = +\infty$, and the solution exists globally in time.

From the Galerkin approximation and the uniform bounds, we pass to the limit to obtain:

$$u \in L^\infty(0,+\infty;H_0^1(\Omega)), \quad u_t \in L^\infty(0,+\infty;L^2(\Omega)) \cap L^{q+1}(\Omega\times(0,+\infty)).$$

By elliptic regularity (since $-\Delta u = u_{tt} + |u_t|^{q-1}u_t + f(u) \in L^2$), we conclude that:

$$u \in L^\infty(0,+\infty;H^2(\Omega)\cap H_0^1(\Omega)).$$

The solution exists for all $t \geq 0$, and satisfies the desired regularity properties.

□

## 2.2 Uniqueness

**Théorème 2.2.1.** *Let* $\mathbb{R}^n$ *and* $n \geq 1$ *be a bounded domain with smooth boundary, and let* $q > 1$*. Assume the nonlinearity* $f : \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous. Then the problem* $(1,1)-(1.2)$ *admits at most one weak solution*

*Proof.* Assume $u_1$ and $u_2$ are two solutions to the equation:

$$u_{tt} - \Delta u + |u_t|^{q-1} u_t + f(u) = 0,$$

with the same initial conditions:

$$u_1(0) = u_2(0) = u_0, \quad \partial_t u_1(0) = \partial_t u_2(0) = u_1.$$

Let $w = u_1 - u_2$, then $w$ satisfies:

$$w_{tt} - \Delta w + \left(|u_{1,t}|^{q-1} u_{1,t} - |u_{2,t}|^{q-1} u_{2,t}\right) + (f(u_1) - f(u_2)) = 0,$$

with initial conditions:

$$w(0) = 0, \quad w_t(0) = 0.$$

Multiply the equation for $w$ by $w_t$ and integrate over $\Omega$:

$$\int_\Omega w_t w_{tt}\, dx - \int_\Omega w_t \Delta w\, dx + \int_\Omega w_t \left(|u_{1,t}|^{q-1} u_{1,t} - |u_{2,t}|^{q-1} u_{2,t}\right) dx + \int_\Omega w_t \left(f(u_1) - f(u_2)\right) dx = 0.$$

Using integration by parts and the boundary condition $w|_{\partial\Omega} = 0$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left(\|w_t\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) + \int_\Omega w_t \left(|u_{1,t}|^{q-1} u_{1,t} - |u_{2,t}|^{q-1} u_{2,t}\right) dx + \int_\Omega w_t \left(f(u_1) - f(u_2)\right) dx = 0.$$

By the Lipschitz continuity of $f$, we get:

$$\left|\int_\Omega w_t \left(f(u_1) - f(u_2)\right) dx\right| \leq L \int_\Omega |w_t||w|\, dx \leq \frac{L}{2}\left(\|w_t\|_{L^2}^2 + \|w\|_{L^2}^2\right).$$

The damping term satisfies:

$$\left(|u_{1,t}|^{q-1} u_{1,t} - |u_{2,t}|^{q-1} u_{2,t}\right)(u_{1,t} - u_{2,t}) \geq 0,$$

due to the monotonicity of the function $s \mapsto |s|^{q-1}s$. Thus:

$$\int_\Omega w_t \left(|u_{1,t}|^{q-1} u_{1,t} - |u_{2,t}|^{q-1} u_{2,t}\right) dx \geq 0.$$

Combining the above estimates, we have:

$$\frac{d}{dt}\left(\|w_t\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) \leq L\left(\|w_t\|_{L^2}^2 + \|w\|_{L^2}^2\right).$$

By the Poincaré inequality, we find:

$$\frac{d}{dt}\left(\|w_t\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) \leq L(1+C)\left(\|w_t\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right).$$

Let $E_w(t) = \|w_t\|_{L^2}^2 + \|\nabla w\|_{L^2}^2$. Then:

$$\frac{dE_w(t)}{dt} \leq L(1+C)E_w(t).$$

By Gronwall's inequality, since $E_w(0) = 0$, we conclude:

$$E_w(t) \leq E_w(0)e^{L(1+C)t} = 0.$$

Thus, $E_w(t) = 0$ for all $t \geq 0$, which implies $w = 0$ and $u_1 = u_2$. □

# Chapter 3

# Long-time dynamics.

To generate a dynamical system associated with the problem (1.1) - (1.2), we first introduce a some notations.

## Notations

We consider the energy space

$$\mathcal{H} = H_0^1(\Omega) \cap H^2(\Omega) \times L^2(\Omega),$$

Let $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

For the family of Hilbert spaces $D(A^{s/2})$, with $s \in \mathbb{R}$, their inner products and norms are respectively defined as:

$$\langle \cdot, \cdot \rangle_{D(A^{s/2})} = \langle A^{s/2} \cdot A^{s/2} \cdot \rangle \quad \| \cdot \|_{D(A^{s/2})} = \|A^{s/2}\|.$$

Thus, we have the inclusion:

$$D(A^{s/2}) \subset D(A^{r/2}) \quad \text{for all } s > r.$$

And the continuous embedding

$$D(A^{s/2}) \hookrightarrow L^{6/(3-2s)}(\Omega) \quad \text{for all } s \in [0, \frac{3}{2}).$$

The following interpolation also holds:
Given $s > r > q'$, for any $\epsilon > 0$, there exists a positive constant $C_\epsilon = C_\epsilon(s, r, q')$ such that

$$|A^{r/2} u| \leq \epsilon |A^{s/2} u| + C_\epsilon |A^{q'/2} u| \quad \text{for all } u \in D(A^{s/2}).$$

Define $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by:

$$\mathcal{A}(u, v) = \left( v, \Delta u - |v|^{q-1} v - f(u) \right).$$

The domain of $\mathcal{A}$ is defined as

$$D(\mathcal{A}) = \left\{ (u, v) \in \mathcal{H} \ : \ |v|^{q-1} v \in L^2, \ f(u) \in L^2 \right\}.$$

It is possible to show that $\mathcal{A}$ generates a continuous semigroup on $\mathcal{H}$.
From chapter 1, for any initial data $U_0 \in \mathcal{H}$, there exists a unique weak solution $U(t) = (u(t), u_t(t))$ of (1.1)-(1.2) satisfying:

$$U \in C([0,T];\mathcal{H}), \quad u \in L^\infty(0,T;V), \quad u_t \in L^\infty(0,T;H) \cap L^q(0,T;L^q(\Omega)).$$

Therefore, the map

$$S(t) : \mathcal{H} \to \mathcal{H}, \quad S(t)(u_0, u_1) = (u(t), u_t(t)),$$

defines a nonlinear dynamical system, or more precisely, a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on the energy space $\mathcal{H}$. This semigroup captures the time evolution of the solution and serves as the foundation for further qualitative analysis, such as the study of global attractors, asymptotic compactness, and long-time behavior.

We need some Assumptions :

1. $f \in C^1(\mathbb{R})$ with $f(0) = 0$ satisfying

$$|f'(r)| \leq C_1(1 + |r|^q), \tag{3.1}$$

$$\liminf_{|r|\to\infty} \frac{f(r)}{r} > -\lambda_1. \tag{3.2}$$

2. $\exists c_1, c_2 > 0$, and $p > 1$ such that

$$|f(s)| \leq c_1(1 + |s|^p),$$

and

$$F(s) \geq -c_2. \quad \forall s \in \mathbb{R}. \tag{3.3}$$

3. $g(u_t) = |u_t|^{q-1}u_t$, $g \in C(\mathbb{R})$ satisfying

$\exists c_1, c_2 > 0 \subseteq T$

$$\forall s \in \mathbb{R}, \qquad c_1|s|^{p+1} \leq g(s) \leq c_2|s|^{p+1}, \tag{3.4}$$

with the natural norm

$$\|(u,v)\|^2_{\mathcal{H}} = \|u\|^2_V + \|v\|^2_{L^2}.$$

We define the variable $U(t) = (u(t), u_t(t)) \in \mathcal{H}$. Then the system (1.1) - (1.2) can be reformulated as a first-order evolution equation in $\mathcal{H}$:

$$\frac{d}{dt}U(t) = \mathcal{A}(U(t)), \quad U(0) = (u_0, u_1). \tag{3.5}$$

## Dissipative

In this subsection we prove the existence of a uniform bounded closed absorbing set of $\{U(t), 0\}$.

**Théorème 3.0.1.** *Suppose that $g$ and $f$ satisfy Assumption (3.1)-(3.4) and $u_1 \neq 0$, then $\{U(t), 0\}$ (3.5) has a bounded uniformly absorbing set $\tilde{\mathcal{B}}$ in $\mathcal{H}$.*
*More precisely, there exist a positive constant $N$, such that for any bounded subset $\mathcal{B} \subset \mathcal{H}$ , there exist a time $T_{\mathcal{B}} > 0$, $T = T(\mathcal{B})$ such that for any $t \geq T_{\mathcal{B}}$, we have*

$$\|U(t,0)\mathcal{B}\|_{\mathcal{H}} \leq N. \quad \forall (u(0) \in \mathcal{B}).$$

*Proof.* with the modified velocity defined as:

$$\tilde{u} = u_t + \epsilon u, \quad \text{where } 0 < \epsilon \ll 1.$$

We define the modified energy functional:

$$\Phi(t) = \|\tilde{u}(t)\|^2 + \|\nabla u(t)\|^2 + 2\int_\Omega F(u)\,dx,$$

and assume that it satisfies the energy identity:

$$\frac{d}{dt}\Phi(t) + 2\Psi(t) = 0. \tag{3.6}$$

Our goal is to explicitly compute the dissipation term $\Psi(t)$.
Since $\tilde{u} = u_t + \epsilon u$, we have:

$$\frac{d}{dt}\|\tilde{u}\|^2 = 2\int_\Omega u_t u_{tt} + \epsilon u u_{tt} + \epsilon u_t^2 + \epsilon^2 u u_t.$$
$$\frac{d}{dt}\|\nabla u\|^2 = 2\int_\Omega \nabla u \cdot \nabla u_t\,dx = -2\int_\Omega \Delta u \cdot u_t\,dx.$$

And we have:

$$\frac{d}{dt}\left(2\int_\Omega F(u)\,dx\right) = 2\int_\Omega f(u)u_t\,dx.$$

Combining all terms together, we have:

$$\frac{d}{dt}\Phi(t) = 2\left[\int_\Omega u_t u_{tt} + \epsilon\int_\Omega u u_{tt} + \epsilon\int_\Omega u_t^2 + \epsilon^2\int_\Omega u u_t - \int_\Omega \Delta u \cdot u_t + \int_\Omega f(u)u_t\right]. \tag{3.7}$$

We simplify:

$$\begin{aligned}\frac{d}{dt}\Phi(t) = 2\Bigg[&-\|\nabla u_t\|^2 - \|u_t\|_{L^{q+1}}^{q+1} - \int_\Omega f(u)u_t \\ &+\epsilon(-\|\nabla u\|^2 - \int_\Omega u|u_t|^{q-1}u_t - \int_\Omega uf(u)) + \epsilon\|u_t\|^2 \\ &+\epsilon^2\int_\Omega uu_t\Bigg] - \int_\Omega \Delta u \cdot u_t + \int_\Omega f(u)u_t\Bigg].\end{aligned}$$

Substitute this formula into (3.7), we get:

$$\begin{aligned}\frac{d}{dt}\Phi(t) = 2\Bigg[&-\|\nabla u_t\|^2 - \|u_t\|_{L^{q+1}}^{q+1} + \epsilon\left(-\|\nabla u\|^2 - \int_\Omega u|u_t|^{q-1}u_t - \int_\Omega uf(u) + \|u_t\|^2\right) \\ &+\epsilon^2\int_\Omega uu_t\Bigg].\end{aligned} \tag{3.8}$$

Then:

$$\begin{aligned}\Psi(t) = \|\nabla u_t\|^2 + \|u_t\|_{L^{q+1}}^{q+1} + \epsilon\left(\|\nabla u\|^2 + \int_\Omega u|u_t|^{q-1}u_t\,dx + \int_\Omega uf(u)\,dx - \|u_t\|^2\right) \\ - \epsilon^2\int_\Omega uu_t\,dx.\end{aligned} \tag{3.9}$$

Indeed, (3.2) implies that for every $\delta > 0$, and constant $C_\delta$:

$$f(u)u \geq -(\lambda_1 - \delta)|u|^2 - C_\delta, \tag{3.10}$$

$$F(u) \geq -\frac{1}{2}(\lambda_1 - \delta)|u|^2 - C_\delta. \tag{3.11}$$

From the assumption on $F(u)$, and using Poincaré inequality, we obtain:

$$2\int_\Omega F(u)\,dx \geq -\frac{(\lambda_1 - \delta)}{\lambda_1}\|\nabla u\|^2 - 2C_\delta|\Omega|.$$

Substitute into $\Phi(t)$ we get:

$$\Phi(t) \geq \|\tilde{u}(t)\|^2 + \frac{\delta}{\lambda_1}\|\nabla u(t)\|^2 - 2C_\delta|\Omega|. \tag{3.12}$$

To estimate $\int_\Omega F(u)\,dx$, we note that:

$$|f(u)| = \left|\int_0^u f'(s)\,ds\right| \leq \int_0^{|u|} |f'(s)|\,ds.$$

Using (3.1), we get:

$$|f(u)| \leq C_1 \int_0^{|u|} (1 + s^q)\,ds = C_1\left(|u| + \frac{1}{q+1}|u|^{q+1}\right).$$

Integrating this formula, we obtain:

$$\int_\Omega |F(u)|\,dx \leq C\int_\Omega \left(|u|^2 + |u|^{q+2}\right)dx.$$

By Sobolev embedding, and Poincaré's inequality we get:

$$\int_\Omega |F(u)|\,dx \leq C\left(\|u\|_{L^2}^2 + \|u\|_{L^{q+2}}^{q+2}\right) \leq C\left(\|\nabla u\|^2 + \|\nabla u\|^{q+2}\right).$$

Combining all terms:

$$\Phi(t) \leq C_f\left(\|\tilde{u}(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla u(t)\|^{q+2}\right).$$

In particular, for $q = 2$ , we get:

$$\Phi(t) \leq C_f(\|\tilde{u}(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla u(t)\|^4). \tag{3.13}$$

Using (3.10), we get:

$$\langle f(u), u\rangle \geq \frac{\lambda_1 - \delta}{\lambda_1}\|\nabla u(t)\|^2 - C_\delta|\Omega|.$$

We suppose that:

$$h(t) = \epsilon\delta\|u(t)\|^2 - 2\epsilon(\epsilon + |u_t|^{q-1})\|u(t)\|\|\tilde{u}(t)\| + (|u_t|^{q-1} - 2\epsilon)\|\tilde{u}(t)\|^2.$$

We use the identity (3.6):

Then dissipation term $\Psi(t)$ can be expressed as:

$$\begin{aligned}\Psi(t) &= \int_\Omega |u_t|^{q-1}\tilde{u}^2\,dx + \epsilon\langle f(u),u\rangle \quad + \epsilon\delta\|u\|^2 - 2\epsilon(\epsilon + |u_t|^{q-1})\|u\|\|\tilde{u}\| - 2\epsilon\|\tilde{u}\|^2. \\ &= (|u_t|^{q-1} - 2\epsilon)\|\tilde{u}(t)\|^2 + \epsilon\langle f(u),u\rangle + \epsilon\delta\|u(t)\|^2 \quad - 2\epsilon(\epsilon + |u_t|^{q-1})\|u(t)\|\|\tilde{u}(t)\| \\ &= \left(|u_t|^{q-1} - 2\epsilon\right)\|\tilde{u}(t)\|^2 + \epsilon\langle f(u),u\rangle + \epsilon\delta\|u(t)\|^2 - 2\epsilon(\epsilon + |u_t|^{q-1})\|u(t)\|\|\tilde{u}(t)\|.\end{aligned}$$

Then we get:

$$\Psi(t) = \left(|u_t|^{q-1} - 2\epsilon\right)\|\tilde{u}(t)\|^2 + \epsilon\langle f(u),u\rangle + h(t) - \left(|u_t|^{q-1} - 2\epsilon\right)\|\tilde{u}(t)\|^2.$$

Using the assumption:

$$\langle f(u),u\rangle \geq \frac{\lambda_1 - \delta}{\lambda_1}\|\nabla u(t)\|^2 - C_\delta|\Omega|,$$

and we assume $\delta \leq \frac{\lambda_1}{2}$, then:

$$\frac{\lambda_1 - \delta}{\lambda_1} \geq \frac{\delta}{2\lambda_1},$$

so:

$$\epsilon\langle f(u),u\rangle \geq \frac{\epsilon\delta}{2\lambda_1}\|\nabla u(t)\|^2 - \epsilon C_\delta|\Omega|.$$

Thus:

$$\Psi(t) \geq (|u_t|^{q-1} - 2\epsilon)\|\tilde{u}(t)\|^2 + \frac{\epsilon\delta}{2\lambda_1}\|\nabla u(t)\|^2 - \epsilon C_\delta|\Omega| + h(t).$$

Assuming $|u_t|^{q-1} \geq 4\epsilon$, then:

$$(|u_t|^{q-1} - 2\epsilon) \geq \frac{1}{2}|u_t|^{q-1},$$

$$(|u_t|^{q-1} - 2\epsilon)\|\tilde{u}(t)\|^2 \geq \frac{1}{2}|u_t|^{q-1}\|\tilde{u}(t)\|^2.$$

Therefore:

$$\Psi(t) \geq \frac{\epsilon\delta}{2\lambda_1}\|\nabla u(t)\|^2 + \frac{1}{2}|u_t|^{q-1}\|\tilde{u}(t)\|^2 + \frac{1}{2}h(t) - C_\delta|\Omega|.$$

We can take $\epsilon_1 = \min\left\{\frac{1}{2}, \frac{(\alpha-2)\delta}{(\beta+1)^2}\right\}$ small enough such that $h(t) \geq 0$, where:

- $\alpha = \inf|u_t(x,t)|^{q-1}$ ,$\alpha > 2$.
- $\beta = \sup|u_t(x,t)|^{q-1}$ .
- $\delta$ is a geometric parameter (typically $\delta = 1$).
- The damping coefficient $|u_t|^{q-1}$ should be bounded: $0 < \alpha \leq |u_t|^{q-1} \leq \beta$.

Then we have

$$\Psi(t) \geq \frac{\epsilon\delta}{2\lambda_1}\|\nabla u(t)\|^2 + \frac{1}{2}\alpha\|\tilde{u}(t)\|^2 - \epsilon C_\delta|\Omega|. \tag{3.14}$$

Substituting (3.14) , and using the energy identity (3.8):
we obtain:

$$\begin{aligned}\frac{d}{dt}\Phi(t) &\leq -2\left(\frac{\epsilon\delta}{2\lambda_1}\|\nabla u(t)\|^2 + \frac{1}{2}\alpha\|\tilde{u}(t)\|^2 - \epsilon C_\delta|\Omega|\right) \\ &= -\frac{\epsilon\delta}{\lambda_1}\|\nabla u(t)\|^2 - \alpha\|\tilde{u}(t)\|^2 + 2\epsilon C_\delta|\Omega|,\end{aligned}$$

then:

$$\frac{d}{dt}\Phi(t)+\frac{\epsilon\delta}{\lambda_1}\|\nabla u(t)\|^2+\frac{1}{2}\alpha\|\tilde{u}(t)\|^2\leq\mu,$$

where $\mu=2\epsilon C_\delta|\Omega|$. let

$$\epsilon=\min\{\epsilon_1,\sqrt{\frac{\lambda_1}{2}}\},\quad \vartheta=\min\{\frac{\epsilon\delta}{2\lambda_1},\frac{\alpha}{4}\},$$

we obtain

$$\frac{d}{dt}\Phi(t)+\vartheta(\|\nabla u(t)\|^2+\|u_t(t)\|^2)\leq\mu. \tag{3.15}$$

Applying Gronwall inequality, and combining with (3.12), (3.13) and (3.15), we get

$$\Phi(U(t))\leq\sup_{U=(u,u_t)\in\mathcal{H}}\{\Phi(U):\vartheta\|U\|^2_{\mathcal{H}}\leq 2\mu\},\quad \forall U(t)=(u(t),u_t(t))\in\mathcal{H},$$

provided that

$$t\geq T_B=\frac{2C_\delta|\Omega|+2C_f(\|B\|^2_{\mathcal{H}}+\|B\|^4_{\mathcal{H}})}{\mu}.$$

So that $B$ be a bounded subset in Banach space $E$.
Using (3.12) and $\epsilon\leq\sqrt{\frac{\lambda_1}{2}}$ again, we obtain that the set

$$\tilde{\mathcal{B}}=\left\{U\in\mathcal{H}:\|U\|^2_{\mathcal{H}}\leq 2M+4C_\delta|\Omega|\right\}, \tag{3.16}$$

is a bounded uniformly absorbing set for the process $\{U(t,0)\}$ corresponding to (3.5). $\square$

**Remarque 3.0.2.**

1. *In order to ensure that the bounded uniformly absorbing set is semi-invariant, our candidate for the absorbing set is*

$$\hat{\mathcal{B}} = \overline{\bigcup_{t \geq T_{\tilde{\mathcal{B}}}} U(t,0)\tilde{\mathcal{B}}} \subset \tilde{\mathcal{B}}, \tag{3.17}$$

   *where $\tilde{\mathcal{B}}$ is from (3.16).*

2. *Integrating inequality (3.15) over $[0,t]$, $t \leq T(\mathcal{B})$, we have*

$$\Phi(t) \leq \Phi(0) + \mu(t).$$

*Recalling (3.5), (3.12) and (3.13), we get*

$$\|U(t,0)\mathcal{B}\|_{\mathcal{H}} \leq \mathcal{Q}_0(\|\mathcal{B}\|_{\mathcal{H}}), \quad \forall t \geq 0, \tag{3.18}$$

*where the monotonic function $\mathcal{Q}_0$ is independent of t .*

### Lipschitz Continuity

**Lemme 3.0.3.** *For all $t \geq 0$, the skew product flow drfined by*

$$S(t)(u,0) = (U(t,0)u, T(t)), \quad (u,0) \in \mathbb{E} = \mathcal{H} \times \mathbb{R}^n,$$

*is Lipschitz continuous on $\mathcal{B}_S = \hat{B} \times \mathbb{R}^n$ i.e., there exists a constant $\alpha_0 > 0$ such that*

$$\|S(t)x - S(t)y\|_{\mathbb{E}} \leq e^{\alpha_0 t}\|x-y\|_{\mathbb{E}}, \quad \forall x, y \in \mathcal{B}_S.$$

*Where $\alpha_0$ depends on $\mathcal{B}_S$,*

*Proof.* Let $x = (u_1, z_1)$, $y = (u_2, z_2) \in \mathcal{B}_S = \hat{B} \times \mathbb{R}^n$. We want to estimate:

$$\|S(t)x - S(t)y\|_{\mathbb{E}} = \|(U(t,0)u_1, T(t)z_1) - (U(t,0)u_2, T(t)z_2)\|_{\mathcal{H}\times\mathbb{R}^n}.$$

Using the product norm on $\mathbb{E}$:

$$\|S(t)x - S(t)y\|_{\mathbb{E}}^2 = \|U(t,0)u_1 - U(t,0)u_2\|_{\mathcal{H}}^2 + \|T(t)z_1 - T(t)z_2\|_{\mathbb{R}^n}^2.$$

Since $\hat{B} \subset \mathcal{H}$ is bounded and the solution operator $U(t,0)$ is assumed to be differentiable or Lipschitz on $\hat{B}$, there exists $\alpha_1 > 0$ such that:

$$\|U(t,0)u_1 - U(t,0)u_2\|_{\mathcal{H}} \leq e^{\alpha_1 t}\|u_1 - u_2\|_{\mathcal{H}}.$$

Similarly, if $T(t)$ is Lipschitz continuous, then there exists $\alpha_2 \geq 0$ such that:

$$\|T(t)z_1 - T(t)z_2\|_{\mathbb{R}^n} \leq e^{\alpha_2 t}\|z_1 - z_2\|_{\mathbb{R}^n}.$$

Putting both estimates together:

$$\|S(t)x - S(t)y\|_{\mathbb{E}}^2 \leq e^{2\alpha_1 t}\|u_1 - u_2\|_{\mathcal{H}}^2 + e^{2\alpha_2 t}\|z_1 - z_2\|_{\mathbb{R}^n}^2.$$

Let $\alpha_0 = \max(\alpha_1, \alpha_2)$. Then we obtain:

$$\|S(t)x - S(t)y\|_{\mathbb{E}} \leq e^{\alpha_0 t}\|x - y\|_{\mathbb{E}}.$$

□

## 3.1 Asymptotic regularity

**Théorème 3.1.1.** *Let $f$ and $g$ satisfy the Assumption (3.1)-(3.4), and assume further $\alpha > 0$. Then the semigroup $\{S(t)\}_{t\geq 0}$ induced by solutions of (1.1)-(1.2) on $\mathbb{E}$ has a bounded uniform exponentially attracting set.*
*More precisely, there exist a bounded set $\mathcal{M} \times \mathbb{R}^n \subset \mathbb{E}^1$, a constant $\alpha > 0$ and a monotonically increasing function $\mathcal{Q}(\cdot)$, such that for any bounded set $\mathcal{B} \subset \mathbb{E}$, it holds*

$$dist_{\mathbb{E}}\left(S(t)\mathcal{B}, \mathcal{M} \times \mathbb{R}^n\right) \leq \mathcal{Q}(\|\mathcal{B}\|_{\mathbb{E}})e^{-\alpha t}, \quad \text{for all } t \geq 0, \tag{3.19}$$

*where $\mathbb{E} = \mathcal{H} \times \mathbb{R}^n$, $\mathbb{E}^1 = H_0^1(\Omega) \times \mathbb{R}^n$, and $\alpha$ is independent of $\mathcal{B}$.*

### 3.1.1 Decomposition of the equation

For the nonlinear function $f$ satisfying Assumption (3.1)-(3.4), we get that $f$ allows the following decomposition $f = f_0 + f_1$, where $f_0, f_1 \in C^1(\mathbb{R})$ and satisfy:

$$|f_0(u)| \leq C|u|^q, \quad \text{for all } u \in \mathbb{R}, \tag{3.20}$$

$$f_0(u)u \geq 0, \quad \text{for all } u \in \mathbb{R}, \tag{3.21}$$

$$|f_1'(u)| \leq C, \quad \text{for all } u \in \mathbb{R}, \tag{3.22}$$

$$\liminf_{|u|\to\infty} \frac{f_1(u)}{u} > -\lambda_1.$$

In order to obtain the regularity estimates later, we decompose the solution $u(t)$ of equation (1.1)-(1.2) with the initial value $(u(0), u_t(0))$ into the sum

$$U(t,0)(u(0), u_t(0)) = Z(t,0)(u(0), u_t(0)) + Y(t,0)(u(0), u_t(0)) \tag{3.23}$$

for all $t \geq 0$ and all $(u(0), u_t(0)) \in \mathcal{H}$, where $Z(t,0)(u(0), u_t(0)) = (z(t), z_t(t))$ and $Y(t,0)(u(0), u_t(0)) = (y(t), y_t(t))$ are the solution of the following equations:

$$\begin{cases} z_{tt} - \Delta z + |u_t|^{q-1} z_t + f_0(z) = 0, \quad x \in \Omega, \\ z|_{\partial\Omega} = 0, \\ z(x,0) = u(0), \quad z_t(x,0) = u_t(0). \end{cases} \tag{3.24}$$

And

$$\begin{cases} y_{tt} - \Delta y + |u_t|^{q-1} y_t + f(u) - f_0(z) = 0, \quad x \in \Omega, \\ y|_{\partial\Omega} = 0, \\ y(x,0) = 0, \quad y_t(x,0) = 0. \end{cases} \tag{3.25}$$

### 3.1.2 The first a priori estimate

In this subsection, we establish some a priori estimates for the solutions of equations (3.24) and (3.25).

**Lemme 3.1.2.** *There exists a positive constant $M$ such that for any $U(t) = (u(t), u_t(t)) \in \mathcal{H}$,*

$$\|(z(t), z_t(t))\|_{\mathcal{H}}^2 + \|(y(t), y_t(t))\|_{\mathcal{H}}^2 \leq M, \quad \forall t \geq 0. \tag{3.26}$$

*Proof.* The function $z(t)$ satisfies (3.24), we define the energy functional

$$E_Z(t) = \frac{1}{2}\|z_t(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla z(t)\|_{L^2}^2 + \int_\Omega F_0(z(t))\,dx, \quad \text{where } F_0(s) = \int_0^s f_0(\sigma)\,d\sigma.$$

Multiplying the equation (3.27) by $z_t$ and integrating over $\Omega$, we get:

$$\frac{d}{dt}E_Z(t) + \int_\Omega |u_t|^{q-1} z_t^2\,dx = 0.$$

Therefore,

$$E_Z(t) + \int_0^t \int_\Omega |u_t|^{q-1} z_t^2\,dx\,ds = E_Z(0).$$

Under the assumptions $f_0(z)z \geq 0$ and $|f_0(z)| \leq C|z|^q$, we have $F_0(z) \geq 0$, so:

$$E_Z(t) \leq E_Z(0) \leq C\left(\|u(0)\|_{H_0^1}^2 + \|u_t(0)\|_{L^2}^2\right).$$

Hence,

$$\|(z(t), z_t(t))\|_{\mathcal{H}}^2 \leq C, \quad \forall t \geq 0. \tag{3.28}$$

The function $y(t)$ satisfies (3.24), we define the energy functional:

$$E_Y(t) = \frac{1}{2}\|y_t(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla y(t)\|_{L^2}^2.$$

Multiplying the equation (3.29) by $y_t$ and integrating over $\Omega$, we obtain:

$$\frac{d}{dt}E_Y(t) + \int_\Omega |u_t|^{q-1} y_t^2\,dx = -\int_\Omega (f(u) - f_0(z))y_t\,dx.$$

Using Young's inequality:

$$\left|\int_\Omega (f(u) - f_0(z))y_t\,dx\right| \leq \frac{1}{2}\int_\Omega |u_t|^{q-1} y_t^2\,dx + C\int_\Omega \frac{|f(u) - f_0(z)|^2}{|u_t|^{q-1}}\,dx.$$

By assumptions$|f(u)| \leq C(1 + |u|^p), \quad |f_0(z)| \leq C|z|^q$, we get:

$$|f(u) - f_0(z)|^2 \leq C(1 + |u|^{2p} + |z|^{2q}).$$

Since $u = z + y$, and $z$ is bounded, while $u(t)$ lies in the absorbing set $\tilde{\mathcal{B}} \subset \mathcal{H}$, the integral is bounded. Therefore:

$$\frac{d}{dt}E_Y(t) \leq C \quad \Rightarrow \quad E_Y(t) \leq Ct.$$

Because the initial energy is zero and $f(u) - f_0(z) \in L^2(0, T; L^2(\Omega))$, Gronwall's inequality gives:

$$\|(y(t), y_t(t))\|_{\mathcal{H}}^2 \leq C, \quad \forall t \geq 0. \tag{3.30}$$

From (3.28) and (3.30) we find:

$$\|(z(t), z_t(t))\|_{\mathcal{H}}^2 + \|(y(t), y_t(t))\|_{\mathcal{H}}^2 \leq M, \quad \forall t \geq 0,$$

for some constant $M > 0$ depending on the initial data. □

**Lemme 3.1.3** (**Energy Decay Estimate**). *Under the following assumptions:*

1. *The system (3.24) generates a solution semigroup* $\{Z(t,0)\}_{t\geq 0}$ *on the energy space* $\mathcal{H}$
2. *The initial data* $(u(0), u_t(0)) \in \mathcal{H}$.
3. *There exists a damping mechanism satisfying:*

$$\int_0^t \|u_t(r)\|^2 dr \geq \gamma \|(u(0), u_t(0))\|^2_{\mathcal{H}}$$

*for some* $\gamma > 0$ *and all* $t \geq t_0 > 0$

*Then the energy decays exponentially:*

$$\|Z(t,0)(u(0), u_t(0))\|^2_{\mathcal{H}} \leq \mathcal{Q}_1(\|(u(0), u_t(0))\|_{\mathcal{H}}) e^{-k_0 t} \quad \forall t \geq 0, \tag{3.31}$$

*where* $k_0 > 0$ *is a constant and* $\mathcal{Q}_1(\cdot)$ *is monotone increasing function*

.

*Proof.* Consider the following Lyapunov function

$$\mathcal{E}(z, z_t) = \frac{1}{2}\|\nabla z\|^2 + 2\theta\langle z, z_t\rangle + \frac{1}{2}\|z_t\|^2 + \int_\Omega F_0(z)dx,$$

where

$$\theta = \min\left\{\frac{\sqrt{\lambda_1}}{4}, \frac{\alpha}{2}, \frac{\lambda_1 \alpha}{2(\beta^2 + \lambda_1)}\right\}, \tag{3.32}$$

$$F_0(u) = \int_0^u f_0(r)dr.$$

From (3.32), and we apply the Cauchy-Schwarz inequality followed by Young's inequality:

$$2\theta(z, z_t) \leq 2\theta\|z\|\|z_t\| \leq \varepsilon\|z\|^2 + \frac{\theta^2}{\varepsilon}\|z_t\|^2.$$

Assuming and Poincaré-type inequality holds , we get:

$$2\theta(z, z_t) \leq \varepsilon C\|\nabla z\|^2 + \frac{\theta^2}{\varepsilon}\|z_t\|^2.$$

Choosing $\varepsilon$ appropriately, using the above, we estimate:

$$\frac{1}{2}\|\nabla z\|^2 + 2\theta(z, z_t) + \frac{1}{2}\|z_t\|^2 \leq \frac{3}{4}(\|\nabla z\|^2 + \|z_t\|^2). \tag{3.33}$$

Using Young's inequality again, we get:

$$\frac{1}{2}\|\nabla z\|^2 + 2\theta(z, z_t) + \frac{1}{2}\|z_t\|^2 \geq \frac{1}{4}(\|\nabla z\|^2 + \|z_t\|^2). \tag{3.34}$$

From (3.33) and (3.34) we find :

$$\frac{1}{4}(\|\nabla z\|^2 + \|z_t\|^2) \leq \frac{1}{2}\|\nabla z\|^2 + 2\theta\langle z, z_t\rangle + \frac{1}{2}\|z_t\|^2 \leq \frac{3}{4}(\|\nabla z\|^2 + \|z_t\|^2).$$

If $(z(t), z_t(t))$ is a solution of (3.24), using (3.21) and (3.32), we have Differentiate $\mathcal{E}(z(t), z_t(t))$ with respect to time:

$$\frac{d}{dt}\mathcal{E}(z(t), z_t(t)) = \langle \nabla z, \nabla z_t\rangle + \langle z_t, z_{tt}\rangle + 2\theta\langle z_t, z_t\rangle + 2\theta\langle z, z_{tt}\rangle + \int_\Omega f_0(z) z_t\, dx.$$

Using the equation $z_{tt} = \Delta z - |u_t|^{q-1} z_t - f_0(z)$, we get

$$\frac{d}{dt}\mathcal{E}(z(t), z_t(t)) = -|u_t|^{q-1}\|z_t\|^2 + 2\theta\|z_t\|^2 - 2\theta\int_\Omega |u_t|^{q-1} z z_t\, dx \\ - 2\theta\int_\Omega f_0(z) z\, dx.$$

Using Cauchy-Schwarz Young's, and Poincaré inequalities, and assuming $|u_t|^{q-1} \geq \alpha$, we obtain:

$$\frac{d}{dt}\mathcal{E}(z(t), z_t(t)) + \theta\|\nabla z(t)\|^2 + \left(\frac{1}{2}|u_0|^{q-1} - \theta\right)\|z_t(t)\|^2 \leq 0.$$

Using (3.20) and (3.26), we can conclude that there exists a $k_0 > 0$ small enough such that

$$\frac{d}{dt}\mathcal{E}(z(t), z_t(t)) + k_0\mathcal{E}(z(t), z_t(t)) \leq 0.$$

Applying Gronwall's inequality and using (3.35), we obtain

$$\|Z(t,0)(u(0), u_t(0))\|^2_{\mathcal{H}} \leq \mathcal{Q}_1(\|(u(0), u_t(0))\|_{\mathcal{H}})e^{-k_0 t} \quad \forall t \geq 0. \tag{3.31}$$

□

The next estimate is about higher regularity of the solution of equation (3.25).

**Lemme 3.1.4.** *There exists* $k_1 > 0$ *such that for any* $t \geq 0$,

$$\|Y(t,0)B_0\|^2_{\mathcal{H}^{\frac{1}{4}}} \leq \mathcal{Q}_2(\|B_0\|_{\mathcal{H}})e^{k_1(t)},$$

*where* $\mathcal{Q}_2(\cdot)$ *is a monotone increasing function.*

*Proof.* Take the inner product of (3.25) with $A^{\frac{1}{4}}y_t$, we deduce

$$\frac{1}{2}\frac{d}{dt}\Big(\|A^{\frac{1}{8}}y(t)\|^2 + \|A^{\frac{5}{8}}y(t)\|^2 + |u(t)|^{p-1}\|A^{\frac{1}{8}}y(t)\|^2\Big) \\ + \Big\langle f(u) - f_0(z), A^{\frac{1}{4}}y(t)\Big\rangle = \langle 0, A^{\frac{1}{4}}y(t)\rangle. \tag{3.36}$$

With the nonlinear term

$$\langle f(u) - f_0(z), A^{\frac{1}{4}}y(t)\rangle = \langle f_1(z), A^{\frac{1}{4}}y(t)\rangle + \langle f(u) - f(z), A^{\frac{1}{4}}y(t)\rangle.$$

Note that

$$\langle f_1(z), A^{\frac{1}{4}}y(t)\rangle = \frac{d}{dt}\langle f_1(z), A^{\frac{1}{4}}y(t)\rangle - \langle f_1'(z), A^{\frac{1}{4}}y_t(t)\rangle.$$

By (3.14), and Hölder's inequality we obtain:

$$\Big|\langle f_1'(z) z_t(t), A^{\frac{1}{4}}y(t)\rangle\Big| \leq C\int_\Omega |z_t(t)||A^{\frac{1}{4}}y(t)| dx \leq C\|z_t(t)\| \cdot \|A^{\frac{1}{4}}y(t)\|, \tag{3.37}$$

and

$$\Big|\langle f_1(z), A^{\frac{1}{4}}y(t)\rangle\Big| \leq C\int_\Omega |z(t)||A^{\frac{1}{4}}y(t)| dx \leq C\|z(t)\| \cdot \|A^{\frac{1}{4}}y(t)\|. \tag{3.38}$$

Secondly,

$$\langle f(u)-f(z),A^{\frac{1}{4}}y(t)\rangle=\frac{d}{dt}\langle f(u)-f(z),A^{\frac{1}{4}}y(t)\rangle \\ -\langle f'(u)z_t(t)-f'(z)z_t(t),A^{\frac{1}{4}}y(t)\rangle-\langle f'(u)y_t(t),A^{\frac{1}{4}}y(t)\rangle. \tag{3.39}$$

and then recall (3.1) and Hölder's inequality to find

$$\left|\langle f'(u)z_t(t)-f'(z)z_t(t),A^{\frac{1}{4}}y(t)\rangle\right|\leq C\int_\Omega |z_t(t)||y(t)|(1+|u(t)|+|z(t)|)A^{\frac{1}{4}}y(t)dx \\ \leq C\|A^{\frac{5}{8}}y(t)\|^2.$$

Similarly, we get

$$\left|\langle f'(u)y_t(t),A^{\frac{1}{4}}y(t)\rangle\right|\leq C\left(\|A^{\frac{5}{8}}y(t)\|^2+\|A^{\frac{1}{8}}y(t)\|^2\right). \tag{3.40}$$

And

$$\left|\langle f(u)-f(z),A^{\frac{1}{4}}y(t)\rangle\right|\leq C\|y(t)\|_{\mathcal{H}^1}\cdot(1+\|u(t)\|^2_{\mathcal{H}^1}+\|z(t)\|^2_{\mathcal{H}^1})\cdot\|A^{\frac{1}{4}}y(t)\|. \tag{3.41}$$

Finally, combining (3.36)–(3.37) and (3.38)–(3.39), we obtain

$$\frac{d}{dt}\left(\|A^{\frac{5}{8}}y(t)\|^2+\|A^{\frac{1}{8}}y(t)\|^2+\mathcal{F}(t)\right)\leq C\left(\|A^{\frac{5}{8}}y(t)\|^2+\|A^{\frac{1}{8}}y(t)\|^2\right), \tag{3.42}$$

Where $\mathcal{F}(t)=2\langle f(u)-f(z),A^{\frac{1}{4}}y(t)\rangle+2\langle f_1(z),A^{\frac{1}{4}}y(t)\rangle$.

Applying Young's inequality in (3.38) and (3.41), we get

$$|\mathcal{F}(t)|\leq\varepsilon\|A^{\frac{5}{8}}y(t)\|^2+C_{\varepsilon M}, \tag{3.43}$$

where $\varepsilon>0$ is sufficiently small. Combining now (3.42) and (3.43), we conclude

$$\frac{d}{dt}\left(\|A^{\frac{5}{8}}y(t)\|^2+\|A^{\frac{1}{8}}y(t)\|^2+\mathcal{F}(t)\right)\leq k_1\left(\|A^{\frac{5}{8}}y(t)\|^2+\|A^{\frac{1}{8}}y(t)\|^2+\mathcal{F}(t)\right)+C_{\varepsilon M}.$$

Where $k_1$ depend on $\|z_t(t)\|$, $\|z(t)\|_{\mathcal{H}^1}$ and $\|u(t)\|_{\mathcal{H}^1}$.
Applying Gronwall's inequality and recalling

$$\|U(t,0)\mathcal{B}\|_{\mathcal{H}}\leq\mathcal{Q}_0(\|\mathcal{B}\|_{\mathcal{H}}),\quad\forall t\geq 0.$$

Using (3.31) and (3.43), we have:

$$\|A^{\frac{5}{8}}y(t)\|^2+\|A^{\frac{1}{8}}y_t(t)\|^2\leq\mathcal{Q}_2(\|(y(0),y_t(0))\|_{\mathcal{H}})e^{-\gamma t}+C_{\varepsilon M}.$$

Finally, since $\|Y(t,0)B_0\|^2_{\mathcal{H}^{\frac{1}{4}}}\sim\|A^{\frac{1}{8}}y_t\|^2+\|A^{\frac{5}{8}}y\|^2$, we conclude that:

$$\|Y(t,0)B_0\|^2_{\mathcal{H}^{\frac{1}{4}}}\leq\mathcal{Q}_2(\|B_0\|_{\mathcal{H}})e^{k_1t}.$$

□

Similarly to [37], [38], and [41], based on Lemma 3.1.3 and Lemma 3.1.4, we can now decompose the solution $u(t)$ of (1.1)-(1.2) as follows

**Lemme 3.1.5.** *Let $(u(t), u_t(t))$ be the solution of (1.1)-(1.2) corresponding to the initial data $(u(0), u_t(0)) \in B_0$. Then, for any $\eta > 0$ there exist positive constants $K_\eta$ and $C_\eta$ such that*

$$u(t) = z_1(t) + y_1(t) \quad \textit{for all } t \geq 0,$$

*where $z_1(t)$ and $y_1(t)$ satisfy the following estimates:*

$$\|y_1(t)\|^2_{\mathcal{H}^{\frac{5}{4}}} \leq K_\eta \quad \textit{for all } t \geq 0$$

*and*

$$\int_s^t \|z_1(r)\|^2_{\mathcal{H}^1} dr \leq \eta(t-s) + C_\eta \quad \textit{for all } t \geq s \geq 0. \tag{3.44}$$

*Proof.* Let $\chi_R : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\chi_R(s) = \begin{cases} 1, & |s| \leq R, \\ 0, & |s| \geq 2R, \end{cases} \quad \text{and} \quad \|\chi'_R\|_\infty \leq C.$$

Define $y_1(t)$ as the solution to the regularized problem:

$$\begin{cases} y_{1,tt} - \Delta y_1 + |y_{1,t}|^{q-1} y_{1,t} + \chi_R(u) f(u) = 0, \\ y_1|_{\partial\Omega} = 0, \\ y_1(0) = u_0, \quad y_{1,t}(0) = u_1. \end{cases}$$

Then define

$$z_1(t) = u(t) - y_1(t).$$

The function $z_1(t)$ satisfies:

$$\begin{cases} z_{1,tt} - \Delta z_1 + \left(|u_t|^{q-1} u_t - |y_{1,t}|^{q-1} y_{1,t}\right) + (f(u) - \chi_R(u) f(u)) = 0, \\ z_1|_{\partial\Omega} = 0, \\ z_1(0) = 0, \quad z_{1,t}(0) = 0. \end{cases} \tag{3.45}$$

Let us analyze the properties of the source term:

- The function $\chi_R(u)$ is smooth, bounded between 0 and 1, and has compact support in the sense that $\chi_R(u(x)) = 0$, whenever $|u(x)| \geq 2R$, and $\chi_R(u(x)) = 1$ when $|u(x)| \leq R$.
- Since $f(u)$ is assumed to have at most polynomial growth, say $|f(u)| \leq C(1 + |u|^p)$, and since $\chi_R(u)$ vanishes when $|u| \geq 2R$, we have:

  $$|\chi_R(u(x)) f(u(x))| \leq C_R \quad \text{for all } x \in \Omega,$$

  for some constant $C_R$ depending on $R$.

  Therefore, $\chi_R(u) f(u) \in L^\infty(\Omega)$ uniformly in time.

Now observe that the equation for $y_1$ is a semilinear wave equation with a globally bounded source term:

$$y_{1,tt} - \Delta y_1 + |y_{1,t}|^{q-1} y_{1,t} = -\chi_R(u) f(u) \in L^\infty(\Omega).$$

From classical regularity theory for damped wave equations (see, [47,48]), we have the following result:

$$\|y_1(t)\|^2_{\mathcal{H}^s} = \|y_1(t)\|^2_{H^s} + \|y_{1,t}(t)\|^2_{H^{s-1}} \le K_\eta, \quad \forall t \ge 0,$$

for all $s < 2$.

Hence, choosing $s = \frac{5}{4} \in (1,2)$, we deduce:

$$\|y_1(t)\|^2_{\mathcal{H}^{5/4}} \le K_\eta, \quad \forall t \ge 0.$$

Define the energy of $z_1$ by

$$E_{z_1}(t) := \frac{1}{2}\left(\|z_{1,t}(t)\|^2 + \|\nabla z_1(t)\|^2\right).$$

Then we using (3.45)

$$\frac{d}{dt}E_{z_1}(t) = -\int_\Omega \left(|u_t|^{q-1}u_t - |y_{1,t}|^{q-1}y_{1,t}\right) z_{1,t}\,dx - \int_\Omega \left(f(u) - \chi_R(u)f(u)\right) z_{1,t}\,dx.$$

By the monotonicity of the function $\phi(s) = |s|^{q-1}s$, we have:

$$\int_\Omega \left(|u_t|^{q-1}u_t - |y_{1,t}|^{q-1}y_{1,t}\right) z_{1,t}\,dx \ge c\|z_{1,t}\|^{q+1}_{L^{q+1}}.$$

Since $f(u) - \chi_R(u)f(u)$ is supported where $|u| \ge R$. If $f(u)$ has polynomial growth, then

$$\left|\int_\Omega \left(f(u) - \chi_R(u)f(u)\right) z_{1,t}\,dx\right| \le \varepsilon\|z_{1,t}\|^2 + \frac{C}{R^\theta}.$$

Using Gronwall-type inequality and Combining the previous steps, we get:

$$\frac{d}{dt}E_{z_1}(t) + c\|z_{1,t}\|^{q+1}_{L^{q+1}} \le \varepsilon\|z_{1,t}\|^2 + \frac{C}{R^\theta}.$$

Integrating over $[s,t]$, and using Poincaré and Young inequalities, we conclude:

$$\int_s^t \|z_1(r)\|^2_{\mathcal{H}^1}\,dr \le \eta(t-s) + C_\eta,$$

by choosing $R$ sufficiently large depending on $\eta$. $\square$

Now we begin to establish the asymptotic regularity of solutions.

**Lemme 3.1.6.** *For all bounded set $B \subset \mathcal{H}$, there exists a positive constant $N_{\|B\|_\mathcal{H}}$ which depends only on the $\mathcal{H}$-bounds of $B$, such that ,*

$$\|Y(t,0)u_0\|^2_{\mathcal{H}^{\frac{1}{4}}} \le N_{\|B\|_\mathcal{H}} \quad \textit{for all } t \ge 0 \textit{ and } u_0 = (u(0), u_t(0)) \in B.$$

*Proof.* We define:

$$\tilde{y}(t) = y_t(t) + \varepsilon y(t).$$

and the modified energy functional by:

$$\mathcal{E}_y(t) = \|\tilde{y}(t)\|^2_{\mathcal{H}^{1/4}} + \|y(t)\|^2_{\mathcal{H}^{5/4}},$$

where $\mathcal{H}^s = D(A^{s/2})$.

We compute:

$$\frac{d}{dt}\mathcal{E}_y(t) = 2\left\langle A^{1/4}\tilde{y}, A^{1/4}\tilde{y}_t\right\rangle + 2\left\langle A^{5/4}y, y_t\right\rangle.$$

Then, we get:

$$\frac{d}{dt}\mathcal{E}_y(t) = 2\left\langle A^{1/4}\tilde{y}, y_{tt}\right\rangle + 2\varepsilon\left\langle A^{1/4}\tilde{y}, y_t\right\rangle + 2\left\langle A^{5/4}y, y_t\right\rangle.$$

From the equation (3.29), we have:

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_y(t) = &-2\langle A^{1/4}\tilde{y}, Ay\rangle - 2\langle A^{1/4}\tilde{y}, |u_t|^{q-1}y_t\rangle + 2\langle A^{1/4}\tilde{y}, f_0(z) - f(u)\rangle \\ &+ 2\varepsilon\langle A^{1/4}\tilde{y}, y_t\rangle + 2\langle A^{5/4}y, y_t\rangle.\end{aligned}$$

Using the identity $A^{1/4}\tilde{y} = A^{1/4}y_t + \varepsilon A^{1/4}y$, then we obtain the identity:

$$\frac{d}{dt}\mathcal{E}_y(t) + \varepsilon\|y(t)\|^2_{\mathcal{H}^{5/4}} + \left(|u_t|^{q-1} - \varepsilon\right)\|\tilde{y}(t)\|^2_{\mathcal{H}^{1/4}} + h_y(t) + 2\left\langle f(u) - f_0(z), A^{1/4}\tilde{y}(t)\right\rangle = 0.$$

We can choose $\varepsilon = \min\left\{\frac{\lambda_1\alpha}{1+\beta}, \frac{\alpha}{2}, \frac{1}{2}\right\}$, we find

$$\frac{d}{dt}\mathcal{E}_y(t) + \epsilon y_y(t) + 2\langle f(u) - f_0(z), A^{\frac{1}{4}}\tilde{y}(t)\rangle \leq \frac{\epsilon}{4}\|\tilde{y}(t)\|^2_{\mathcal{H}^{\frac{1}{4}}}. \tag{3.46}$$

Indeed, we have

$$\begin{aligned}\langle f(u) - f_0(z), A^{\frac{1}{4}}\tilde{y}(t)\rangle = &\frac{d}{dt}\langle f(u) - f(z), A^{\frac{1}{4}}y(t)\rangle + \epsilon\langle f(u) - f(z), A^{\frac{1}{4}}y(t)\rangle \\ &+ \frac{d}{dt}\langle f_1(z), A^{\frac{1}{4}}y(t)\rangle + \epsilon\langle f_1(z), A^{\frac{1}{4}}y(t)\rangle - \mathcal{I}_1 - \mathcal{I}_2 - \mathcal{I}_3,\end{aligned} \tag{3.47}$$

where

$$\begin{aligned}\mathcal{I}_1 &= \langle f'(u)z_t(t) - f'(z)z_t(t), A^{\frac{1}{4}}y(t)\rangle, \\ \mathcal{I}_2 &= \langle f'(u)y_t(t), A^{\frac{1}{4}}y(t)\rangle, \\ \mathcal{I}_3 &= \langle f_1(z)z_t(t), A^{\frac{1}{4}}y(t)\rangle.\end{aligned}$$

We will estimate the integrals $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$.
Due to Assumption (3.1)-(3.4) and Lemma 3.1.6, we get

$$\begin{aligned}|\mathcal{I}_1| &= \left|\langle f'(u)z_t(t) - f'(z)z_t(t), A^{\frac{1}{4}}y(t)\rangle\right| \\ &\leq C\int_\Omega |z_t(t)|y(t)|(1 + |u(t)| + |z(t)|)A^{\frac{1}{4}}y(t)|dx \\ &\leq C\int_\Omega |z_t(t)|y(t)|(1 + |z_1(t)| + |y_1(t)| + |z(t)|)A^{\frac{1}{4}}y(t)|dx.\end{aligned} \tag{3.48}$$

Now, using Hölder's and Sobolev-type inequality, we have

$$\begin{aligned}\int_\Omega |z_t(t)||y(t)||A^{\frac{1}{4}}y(t)|dx &\leq C_{\|z_t(t)\|,\|y(t)\|_{\mathcal{H}^1}} \|y(t)\|_{\mathcal{H}^{\frac{5}{4}}},\\ \int_\Omega |z_t(t)||y(t)||z_1(t)||A^{\frac{1}{4}}y(t)|dx &\leq C_{\|z_t(t)\|} \|\|z_1(t)\|_{\mathcal{H}^1}\|y(t)\|^2_{\mathcal{H}^{\frac{5}{4}}},\\ \int_\Omega |z_t(t)||y(t)||y_1(t)||A^{\frac{1}{4}}y(t)|dx &\leq C_{\|z_t(t)\|,\|y(t)\|_{\mathcal{H}^1}} K_\eta\|y(t)\|_{\mathcal{H}^{\frac{5}{4}}},\\ \int_\Omega |z_t(t)||y(t)||z(t)||A^{\frac{1}{4}}y(t)|dx &\leq C_M\|z_t(t)\|_{\mathcal{H}^1}\|y(t)\|^2_{\mathcal{H}^{\frac{5}{4}}}.\end{aligned}$$

Thanks to Lemma 3.1.3, we can take $T_1$ large enough so that

$$\|z(t)\|_{\mathcal{H}^1} \leq \frac{\mathcal{E}}{C} \quad \text{for all } t \geq T_1. \tag{3.49}$$

Combining (3.48)-(3.49), using Young's inequality, we find

$$|\mathcal{I}_1| \leq \frac{\epsilon}{24}\|y(t)\|^2_{\mathcal{H}^{\frac{5}{4}}} + C_{\|z_t(t)\|,\epsilon}\|z_1(t)\|^2_{\mathcal{H}^1} \cdot \|y(t)\|^2_{\mathcal{H}^{\frac{5}{4}}} + C_{\epsilon K_{\eta,M}}. \tag{3.50}$$

Similarly, we have

$$|\mathcal{I}_2| = \left|\langle f'(u)y_t(t), A^{\frac{1}{4}}y(t)\rangle\right| \leq \int_\Omega |y_t(t)|(1+|y_1(t)|^2+|z_1(t)|^2)|A^{\frac{1}{4}}y(t)|dx, \tag{3.51}$$

and

$$\begin{aligned}\int_\Omega |y_t(t)|\,|A^{\frac{1}{4}}y(t)|\,dx &\leq C_{\|y_t(t)\|}\,\|y(t)\|_{\mathcal{H}^{\frac{5}{4}}},\\ \int_\Omega |y_t(t)|\,|y_1(t)|^2|A^{\frac{1}{4}}y(t)|\,dx &\leq C_{\|y_t(t)\|,K_\eta}\,\|y(t)\|_{\mathcal{H}^{\frac{5}{4}}},\\ \int_\Omega |y_t(t)|\,|z_1(t)|^2|A^{\frac{5}{4}}y(t)|\,dx &\leq \int_\Omega (|\tilde{y}(t)|+\epsilon|y(t)|)\,|z_1(t)|^2|A^{\frac{1}{4}}y(t)|\,dx\\ &\leq C_{\|z_1(t)\|_{\mathcal{H}^1},\|y(t)\|_{\mathcal{H}^1}}\,\|y(t)\|_{\mathcal{H}^{\frac{5}{4}}}\\ &\quad + C_\epsilon\|z_1(t)\|^2_{\mathcal{H}^1}\cdot\|y(t)\|^2_{\mathcal{H}^{\frac{5}{4}}} + \frac{\epsilon}{24}\|\tilde{y}(t)\|^2_{\mathcal{H}^{\frac{1}{4}}}.\end{aligned} \tag{3.52}$$

Combining (3.51)-(3.52), we get

$$|\mathcal{I}_2| \leq \frac{\epsilon}{24}\mathcal{E}_y(t) + C\|z_1(t)\|^2_{\mathcal{H}^1}\cdot\|y(t)\|^2_{\mathcal{H}^{\frac{5}{4}}} + C_0, \tag{3.53}$$

where $C_0$ may depend on $\epsilon, K_\eta, \|y_t(t)\|, \|z_1(t)\|_{\mathcal{H}^1}, \|y(t)\|_{\mathcal{H}^1}$.
Using (3.22), we have

$$|\mathcal{I}_3| \leq \frac{\epsilon}{24}\|y(t)\|^2_{\mathcal{H}^{\frac{5}{4}}} + C_{\epsilon,\|z_t(t)\|}. \tag{3.54}$$

Finally, substituting the estimates (3.47), (3.50), (3.53) and (3.54) into (3.46) and recalling (3.43), we deduce

$$\frac{d}{dt}\mathbf{E}_y(t) + \iota(t)\mathbf{E}_y(t) \leq C_2,$$

where

$$\mathbf{E}_y(t) = \mathcal{E}_y(t) + 2\langle f(u) - f(z), A^{\frac{1}{4}}y(t)\rangle + 2\langle f_1(z), A^{\frac{1}{4}}y(t)\rangle,$$

$$\iota(t) = \frac{\epsilon}{4} - C\|z_1(t)\|^2_{\mathcal{H}^1}.$$

And $C_1$ depends on $\epsilon, \|z_t(t)\|$, and $C_2$ depends on $\epsilon, K_\eta, M$.

Then using the Gronwall's inequality and integrating over $[T_1, t]$, we obtain that

$$\mathbf{E}_y(t) \leq e^{-\int_{T_1}^t \iota(r)dr}\mathbf{E}_y(T_1) + C_2\int_{T_1}^t e^{-\int_s^t \iota(r)dr}ds. \tag{3.55}$$

Taking $\eta$ in (3.44) small enough, we obtain

$$e^{-\int_{T_1}^t \iota(r)dr} \leq e^{-\frac{\epsilon}{8}(t-T_1)}e^{C_\eta C_1}. \tag{3.56}$$

and

$$\int_{T_1}^t e^{-\int_s^t \iota(r)dr}ds \leq e^{C_\eta C_1}\int_{T_1}^t e^{-\frac{\epsilon}{8}(t-s)}ds \leq 8\epsilon^{-1}e^{C_\eta C_1}.$$

Substituting (3.56) into (3.55), we deduce that, for all $t \geq T_1$,

$$\mathbf{E}_y(t) \leq e^{C_\eta C_1}\mathbf{E}_v(T_1) + 8\epsilon^{-1}C_2 e^{C_\eta C_1}.$$

where $C_\eta$, $C_1$, and $C_2$ are constants independent of the initial data, and $T_1 > 0$ is fixed. This implies that $\mathbf{E}_y(t)$ is uniformly bounded for all $t \geq T_1$.

Moreover, since $B \subset \mathcal{H}$ is bounded and $t \mapsto \mathbf{E}_y(t)$ is continuous, there exists a constant $C_{B,T_1}$ such that:

$$\mathbf{E}_y(t) \leq C_{B,T_1}, \quad \forall t \in [0, T_1].$$

Combining the above, we conclude that $\mathbf{E}_y(t)$ is uniformly bounded on $[0, +\infty)$, i.e.,

$$\mathbf{E}_y(t) \leq C = \max\left\{C_{B,T_1},\ e^{C_\eta C_1}\mathbf{E}_y(T_1) + 8\varepsilon^{-1}C_2 e^{C_\eta C_1}\right\}, \quad \forall t \geq 0.$$

In particular, this implies:

$$\|\tilde{y}(t)\|^2_{\mathcal{H}^{1/4}} \leq C, \quad \text{and} \quad \|y(t)\|^2_{\mathcal{H}^{5/4}} \leq C.$$

Since $y_t(t) = \tilde{y}(t) - \varepsilon y(t)$, we estimate:

$$\|y_t(t)\|_{\mathcal{H}^{-3/4}} \leq \|\tilde{y}(t)\|_{\mathcal{H}^{1/4}} + \varepsilon\|y(t)\|_{\mathcal{H}^{1/4}} \leq C + \varepsilon C',$$

because $\|y(t)\|_{\mathcal{H}^{1/4}}$ is bounded via interpolation between $\mathcal{H}$ and $\mathcal{H}^{5/4}$, both of which are bounded.

Thus, we obtain the estimate:

$$\|Y(t,0)u_0\|^2_{\mathcal{H}^{1/4}} = \|y(t)\|^2_{\mathcal{H}^{1/4}} + \|y_t(t)\|^2_{\mathcal{H}^{-3/4}} \leq N_{\|B\|_{\mathcal{H}}},$$

where $N_{\|B\|_{\mathcal{H}}}$ depends only on the bound of $B$ in $\mathcal{H}$.

□

### 3.1.3 The second a priori estimate

**Lemme 3.1.7.** *Assume* $B_\zeta$ *is an bounded set in* $\mathcal{H}^{\frac{1}{4}}$. *Then there exists a positive constant* $M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}}$ *which depends only on the* $\mathcal{H}^{\frac{1}{4}}$*-bounds of* $B_\zeta$ *such that*

$$\|U(t,0)u_0\|^2_{\mathcal{H}^{\frac{1}{4}}} \leq M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}} \quad \text{for all } t \geq 0 \text{ and } u_0 \in B_\zeta.$$

*Proof.* Let $u(t)$ be the solution of equation (1.1)-(1.2) with the initial value $u_0 = (u(0), u_t(0)) \in B_\zeta$. Taking the inner product of (1.1) with $A^{\frac{1}{4}}\tilde{u}$, we obtain

$$\frac{d}{dt}\mathcal{E}_y(t) + \epsilon\|u(t)\|^2_{\mathcal{H}^{\frac{5}{4}}} + |u_t|^{q-1} - \epsilon)\|\tilde{u}(t)\|^2_{\mathcal{H}^{\frac{1}{4}}} + h_u(t) + 2\langle f(u), A^{\frac{1}{4}}\tilde{u}(t)\rangle = 2\langle 0, A^{\frac{1}{4}}\tilde{u}(t)\rangle,$$

where

$$\begin{aligned}
&\mathcal{E}_u(t) = \|\tilde{u}(t)\|^2_{\mathcal{H}^{\frac{1}{4}}} + \|u(t)\|^2_{\mathcal{H}^{\frac{5}{4}}},\\
&\tilde{u}(t) = u_t(t) + \epsilon u(t),\\
&h_u(t) = \epsilon\|u(t)\|^2_{\mathcal{H}^{\frac{5}{4}}} + |u_t|^{q-1} - \epsilon)\|\tilde{u}(t)\|^2_{\mathcal{H}^{\frac{1}{4}}} + 2\epsilon^2\langle u(t), A^{\frac{1}{4}}\tilde{u}(t)\rangle - 2\epsilon|u_t|^{q-1}\langle u(t), A^{\frac{1}{4}}\tilde{u}(t)\rangle.
\end{aligned}$$

Since the other terms can be handled with a similar argument as in the Lemma 3.1.6, we only deal with the nonlinear term. Indeed, we have

$$\langle f(u), A^{\frac{1}{4}}\tilde{u}(t)\rangle = \frac{d}{dt}\langle f(u), A^{\frac{1}{4}}u(t)\rangle - \langle f(u), A^{\frac{1}{4}}u_t(t)\rangle - f'(u)u_t(t), A^{\frac{1}{4}}u(t)\rangle.$$

And

$$|\langle f'(u)u_t(t), A^{\frac{1}{4}}u(t)\rangle| \leq C\int_\Omega |u_t(t)|(1 + |y_1(t)|^2 + |z_1(t)|^2)|A^{\frac{1}{4}}u(t)|dx. \tag{3.57}$$

By Lemma 3.1.5 and using Hölder's and Sobolev-type inequality, we have

$$\begin{aligned}
\int_\Omega |u_t(t)|\,|A^{\frac{1}{4}}u(t)|\,dx &\leq C\|u_t(t)\| \cdot \|u(t)\|_{\mathcal{H}^{\frac{5}{4}}},\\
\int_\Omega |u_t(t)|\,|y_1(t)|^2\,|A^{\frac{1}{4}}u(t)|\,dx &\leq \|u_t(t)\| \cdot \|y_1(t)\|^2_{L^8} \cdot \|A^{\frac{1}{4}}u(t)\|_{L^4}\\
&\leq C_{\|u_t(t)\|, K_\eta} \cdot \|u(t)\|_{\mathcal{H}^{\frac{5}{4}}},\\
\int_\Omega |u_t(t)|\,|z_1(t)|^2\,|A^{\frac{1}{4}}u(t)|\,dx &\leq \int_\Omega (|\tilde{u}(t)| + \epsilon|u(t)|)\,|z_1(t)|^2\,|A^{\frac{1}{4}}u(t)|\,dx\\
&\leq \epsilon\|z_1(t)\|^2_{L^6} \cdot \|u(t)\|_{L^{\frac{12}{5}}} \cdot \|A^{\frac{1}{4}}u(t)\|_{L^4}\\
&\quad + \|z_1(t)\|^2_{L^6} \cdot \|A^{\frac{1}{4}}u(t)\|_{L^4} \cdot \|\tilde{u}(t)\|_{L^{\frac{12}{5}}}\\
&\leq C_{\|z_1(t)\|_{\mathcal{H}^1}, \|u(t)\|_{\mathcal{H}^1}} \cdot \|u(t)\|_{\mathcal{H}^{\frac{5}{4}}}\\
&\quad + C_\epsilon\|z_1(t)\|^2_{\mathcal{H}^1} \cdot \|u(t)\|^2_{\mathcal{H}^{\frac{5}{4}}} + \frac{\epsilon}{8}\|\tilde{u}(t)\|^2_{\mathcal{H}^{\frac{1}{4}}}.
\end{aligned} \tag{3.58}$$

Then

$$
\begin{aligned}
|\langle f(u), A^{\frac{1}{4}}u(t)\rangle| &\le C\int_{\Omega}\left(1+|u(t)|^3\right)A^{\frac{1}{4}}u(t)\,dx \\
&\le C_{\|u(t)\|_{\mathcal{H}^1}}\cdot\|u(t)\|^{\frac{5}{2}}_{\mathcal{H}^{\frac{1}{4}}}.
\end{aligned}
\tag{3.59}
$$

We combine (3.57)-(3.58) and use Young's inequality, we get

$$
|\langle f'(u)u_t(t), A^{\frac{1}{4}}u(t)\rangle| \le \frac{\epsilon}{8}\|u_t(t)\|^2_{\mathcal{H}^{\frac{5}{4}}} + C_{\|z_1(t)\|,\epsilon}\|z_1(t)\|^2_{\mathcal{H}^1}\cdot\|u(t)\|^2_{\mathcal{H}^{\frac{5}{2}}} + C_{\epsilon,K_\eta,M}.
$$

Therefore, by choosing $\epsilon$ small enough and applying (3.59), we obtain

$$
\frac{d}{dt}\mathcal{E}_u(t) + \zeta(t)\mathcal{E}_u(t) \le C_{\epsilon,K_\eta,M}, \tag{3.60}
$$

where

$$
\mathcal{E}_u(t) = \mathcal{E}_u(t) + 2\langle f(u), A^{\frac{1}{4}}u(t)\rangle,
$$

$$
\zeta(t) = \frac{\epsilon}{4} - C_{\epsilon,\|z_1(t)\|}\|z_1(t)\|^2_{\mathcal{H}^1}.
$$

Multiply both sides of (3.60) by the $e^{\displaystyle\int_0^t \zeta(r)\,dr}$:

$$
\frac{d}{dt}\left(e^{\int_0^t \zeta(r)\,dr}\mathcal{E}_u(t)\right) \le C_{\epsilon,K_\eta,M}e^{\int_0^t \zeta(r)\,dr}.
$$

Integrating from 0 to $t$, we obtain:

$$
\mathcal{E}_u(t) \le e^{-\int_0^t \zeta(r)\,dr}\mathcal{E}_u(0) + C_{\epsilon,K_\eta,M}\int_0^t e^{-\int_s^t \zeta(r)\,dr}\,ds.
$$

By $u_0 = (u(0), u_t(0))$ belongs to $\mathcal{H}^{\frac{1}{4}}$, we then complete the proof by the same argument as in Lemma 3.1.6.

□

As a consequence of Lemma 3.1.6 and 3.1.7, we can state the following result.

**Corollaire 3.1.8.** *Assume $B_\zeta$ is an arbitrary bounded set in $\mathcal{H}^{\frac{1}{4}}$. Then there exists a positive constant $M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}}$ which depends only on the $\mathcal{H}^{\frac{1}{4}}$-bounds of $B_\zeta$ such that*

$$
\|Z(t,0)u_0\|^2_{\mathcal{H}^{\frac{1}{4}}} \le M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}} \quad \textit{for all} \quad t \ge 0 \textit{ and } u_0 = (u(0), u_t(0)) \in B_\zeta.
$$

*Proof.* Let $u(t) = z(t) + y(t)$ be the decomposition of the full solution associated to the system, where $Z(t,0)u_0 = z(t)$ and $Y(t,0)u_0 = y(t)$.

From Lemma 3.1.6, we know that for any bounded set $B \subset \mathcal{H}$, there exists a constant $N_{\|B\|_{\mathcal{H}}} > 0$ such that

$$
\|Y(t,0)u_0\|^2_{\mathcal{H}^{\frac{1}{4}}} \le N_{\|B\|_{\mathcal{H}}}, \quad \forall t \ge 0.
$$

From Lemma 3.1.7, for a bounded set $B_\zeta \subset \mathcal{H}^{\frac{1}{4}}$, there exists a constant $M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}} > 0$ such that

$$
\|U(t,0)u_0\|^2_{\mathcal{H}^{\frac{1}{4}}} \le M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}}, \quad \forall t \ge 0.
$$

Since the total solution satisfies $u(t) = Z(t,0)u_0 + Y(t,0)u_0$, we can write

$$Z(t,0)u_0 = u(t) - Y(t,0)u_0.$$

Therefore,

$$\|Z(t,0)u_0\|_{\mathcal{H}^{\frac{1}{4}}} \leq \|u(t)\|_{\mathcal{H}^{\frac{1}{4}}} + \|Y(t,0)u_0\|_{\mathcal{H}^{\frac{1}{4}}}.$$

Applying the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain

$$\|Z(t,0)u_0\|^2_{\mathcal{H}^{\frac{1}{4}}} \leq 2\|u(t)\|^2_{\mathcal{H}^{\frac{1}{4}}} + 2\|Y(t,0)u_0\|^2_{\mathcal{H}^{\frac{1}{4}}}.$$

Using the uniform bounds on $\|u(t)\|^2_{\mathcal{H}^{1/4}}$ from Lemma 3.1.7 and $\|Y(t,0)u_0\|^2_{\mathcal{H}^{1/4}}$ from Lemma 3.1.6, we conclude that there exists a constant $M_{\|B_\zeta\|_{\mathcal{H}^{1/4}}} > 0$ such that

$$\|Z(t,0)u_0\|^2_{\mathcal{H}^{\frac{1}{4}}} \leq M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}}, \quad \forall t \geq 0.$$

This completes the proof. □

**Lemme 3.1.9.** *For all bounded set $B_\zeta \subset \mathcal{H}^{\frac{1}{4}}$, there exists a positive constant $M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}}$ which depends only on the $\mathcal{H}^{\frac{1}{4}}$-bounds of $B_\zeta$ such that*

$$\|Y(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}} \leq M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}} \quad \text{for all} \quad t \geq 0 \text{ and } u_0 = (u(0), u_t(0)) \in B_\zeta.$$

*Proof.* Taking the inner product of (3.25) with $A^{\frac{3}{4}}\tilde{y}$, we obtain

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}_\nu(t) + \epsilon\|y(t)\|^2_{H^{\frac{7}{4}}} + (|u_t|^{q-1} - \epsilon)\|\tilde{y}(t)\|^2_{H^{\frac{3}{4}}} \\ &+ h_\nu(t) + 2\langle f(u) - f_0(z), A^{\frac{3}{4}}\tilde{y}(t)\rangle = 2\langle 0, A^{\frac{3}{4}}u(t)\rangle, \end{aligned} \tag{3.61}$$

where

$$\begin{aligned} \mathcal{E}_\nu(t) &= \|\tilde{y}(t)\|^2_{\mathcal{H}^{\frac{3}{4}}} + \|y(t)\|^2_{\mathcal{H}^{\frac{7}{4}}}, \quad \tilde{y}(t) = y(t) + \epsilon y(t), \\ h_\nu(t) &= \epsilon\|y(t)\|^2_{\mathcal{H}^{\frac{7}{4}}} + (|u_t|^{q-1}) - \epsilon)\|\tilde{y}(t)\|^2_{\mathcal{H}^{\frac{3}{4}}} \\ &\quad + 2\epsilon^2\langle y(t), A^{\frac{3}{4}}\tilde{y}(t)\rangle - 2\epsilon|u_t|^{q-1}\langle y(t), A^{\frac{3}{4}}\tilde{y}(t)\rangle. \end{aligned}$$

The nonlinear term gives

$$\begin{aligned} \langle f(u) - f_0(z), A^{\frac{3}{4}}\tilde{y}(t)\rangle = &\frac{d}{dt}\langle f(u) - f_0(z), A^{\frac{3}{4}}y(t)\rangle + \epsilon\langle f(u) - f_0(z), A^{\frac{3}{4}}y(t)\rangle \\ &- \langle f'(u)u_t(t) - f_0'(z)z_t(t), A^{\frac{3}{4}}y(t)\rangle. \end{aligned} \tag{3.62}$$

And

$$\left|\langle f'(u)u_t(t), A^{\frac{3}{4}}v(t)\rangle\right| \leq C\int_\Omega |u_t(t)|(1 + |u(t)|^2)|A^{\frac{3}{4}}v(t)|dx.$$

By Lemma 3.1.7, we can deduce

$$\begin{aligned} \int_\Omega |u_t(t)||u(t)|^2|A^{\frac{3}{4}}y(t)|dx &\leq \|u_t(t)\|_{L^{\frac{12}{5}}} \cdot \|u(t)\|^2_{L^{12}} \cdot \|A^{\frac{3}{4}}y(t)\|_{L^{\frac{12}{5}}} \\ &\leq C_{\|u_t(t)\|_{\mathcal{H}^{\frac{1}{4}}}, \|u(t)\|_{\mathcal{H}^{\frac{5}{4}}}}\|y_t(t)\|_{\mathcal{H}^{\frac{7}{4}}}. \end{aligned}$$

Similarly, thanks to Corollary 3.1.8, we get

$$|\langle f_0(z)z_t(t), A^{\frac{3}{4}}y(t)\rangle| \leq C_{|\Omega|,\|z_t(t)\|_{\mathcal{H}^{\frac{1}{4}}},\|z(t)\|_{\mathcal{H}^{\frac{5}{4}}}}\|y_t(t)\|_{\mathcal{H}^{\frac{7}{4}}}. \tag{3.63}$$

Substituting (3.62) and (3.63) into (3.61), and using Young's inequality, we obtain

$$\frac{d}{dt}\mathcal{E}_\nu(t) + \epsilon\mathcal{E}_\nu(t) \leq C(t),$$

where $C(t) \leq C_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}}$ is uniformly bounded for $u_0 \in B_\zeta \subset \mathcal{H}^{\frac{1}{4}}$, we obtain

$$\mathcal{E}_\nu(t) \leq \mathcal{E}_\nu(0)e^{-\epsilon t} + \int_0^t C(s)e^{-\epsilon(t-s)}ds.$$

Since $\mathcal{E}_\nu(0) \leq C_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}}$, we conclude

$$\mathcal{E}_\nu(t) \leq M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}}, \quad \text{for all } t \geq 0.$$

Hence,

$$\|Y(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}} \leq M_{\|B_\zeta\|_{\mathcal{H}^{\frac{1}{4}}}}, \quad \text{for all } t \geq 0,\ u_0 \in B_\zeta.$$

□

**Lemme 3.1.10.** *For all bounded set $B_\nu \subset \mathcal{H}^{\frac{3}{4}}$, there exists a positive constant $M_{\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}}$ which depends only on the $\mathcal{H}^{\frac{3}{4}}$-bounds of $B_\nu$ such that*

$$\|U(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}} \leq M_{\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}} \quad \textit{for all } t \geq 0 \textit{ and } u_0 = (u(0), u(0)) \in B_\nu. \tag{3.64}$$

*Proof.* We consider the solution $u(t)$ of the damped nonlinear wave equation with initial data $u_0 \in B_\nu \subset \mathcal{H}^{\frac{3}{4}}$.

Let us define the higher-order energy functional:

$$\mathcal{E}(t) = \|u_t(t)\|^2_{H^{\frac{3}{4}}} + \|u(t)\|^2_{H^{\frac{7}{4}}}.$$

Taking the time derivative and using the equation, we obtain :

$$\frac{d}{dt}\mathcal{E}(t) + c_1\|u_t(t)\|^{q+1}_{H^{\frac{3}{4}}} + c_2\|u(t)\|^2_{H^{\frac{7}{4}}} \leq \left|\langle f(u), A^{\frac{3}{4}}u_t(t)\rangle\right|.$$

Using the growth condition on $f(u)$ and Sobolev embeddings, we estimate:

$$\left|\langle f(u), A^{\frac{3}{4}}u_t\rangle\right| \leq C\int_\Omega |u_t|(1+|u|^p)|A^{\frac{3}{4}}u|\,dx \leq C_{\|u\|_{H^{\frac{7}{4}}},\|u_t\|_{H^{\frac{3}{4}}}}.$$

Thus,

$$\frac{d}{dt}\mathcal{E}(t) + \epsilon\mathcal{E}(t) \leq C(t),$$

where $C(t)$ depends only on norms of the solution, which are uniformly bounded for $u_0 \in B_\nu$.

Applying Gronwall's lemma, we get:

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-\epsilon t} + \int_0^t C(s)e^{-\epsilon(t-s)}ds \leq M_{\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}}.$$

Hence,

$$\|U(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}} \leq M_{\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}}, \quad \text{for all } t \geq 0,\ u_0 \in B_\nu.$$

□

As an immediate consequence of the Lemma 3.1.9 and 3.1.10, we have the following result:

**Corollaire 3.1.11.** *For all bounded set $B_\nu \subset \mathcal{H}^{\frac{3}{4}}$, there exists a positive constant $M_{\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}}$ which depends only on the $\mathcal{H}^{\frac{3}{4}}$-bounds of $B_\nu$ such that*

$$\|Z(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}} \leq M_{\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}} \quad \textit{for all } t \geq 0 \textit{ and } u_0 = (u(0), u(0)) \in B_\nu.$$

*Proof.* From the decomposition of the evolution operator, we have

$$U(t,0)u_0 = Z(t,0)u_0 + Y(t,0)u_0.$$

Taking norms in $\mathcal{H}^{\frac{3}{4}}$, we apply the triangle inequality:

$$\|Z(t,0)u_0\|_{\mathcal{H}^{\frac{3}{4}}} \leq \|U(t,0)u_0\|_{\mathcal{H}^{\frac{3}{4}}} + \|Y(t,0)u_0\|_{\mathcal{H}^{\frac{3}{4}}}.$$

Applying the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain:

$$\|Z(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}} \leq 2\|U(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}} + 2\|Y(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}}.$$

Now, by Lemma 3.1.10 and Lemma 3.1.9 , there exist constants depending only on $\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}$ such that

$$\|U(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}} \leq M_1, \quad \|Y(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}} \leq M_2.$$

Therefore,

$$\|Z(t,0)u_0\|^2_{\mathcal{H}^{\frac{3}{4}}} \leq 2M_1 + 2M_2 =: M_{\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}}.$$

This completes the proof. □

**Lemme 3.1.12.** *Assume the initial data set $B_\nu$ is bounded in $\mathcal{H}^{\frac{3}{4}}$ and $\frac{3}{4} \leq \varrho \leq 1$, then the solution of (3.25) $y(t)$ satisfies that,*

$$\|Y(t,0)u_0\|^2_{\mathcal{H}^\varrho} \leq N_{\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}} \quad \textit{for all } t \geq 0 \textit{ and } u_0 = (u(0), u(0)) \in B_\nu.$$

*Proof.* We test equation (3.25) with $A^\varrho \tilde{y}(t)$, where $\tilde{y}(t) = y_t(t) + \epsilon y(t)$, for a small parameter $\epsilon > 0$. By multiplying the equation (3.25) by this test function and integrating by parts, we obtain:

$$\frac{d}{dt}\mathcal{E}_\varrho(t) + \epsilon\mathcal{E}_\varrho(t) \leq 2\langle f'(u)u_t(t), A^\varrho y(t)\rangle + 2\langle f_0'(z)z_t(t), A^\varrho y(t)\rangle, \tag{3.65}$$

where :

$$\mathcal{E}_\varrho(t) = \|\tilde{y}(t)\|^2_{\mathcal{H}^\varrho} + \|y(t)\|^2_{\mathcal{H}^{\varrho+1}} + 2\langle f(u) - f_0(z), A^\varrho y(t)\rangle.\tilde{y}(t) = y(t) + \epsilon y(t).$$

We now estimate the nonlinear terms on the right-hand side of (3.65). Using Lemma 3.1.10 and the continuous embedding $\mathcal{H}^{\frac{7}{4}} \hookrightarrow L^\infty$, we apply Hölder's inequality to get:

$$\begin{aligned} 2\left|\langle f'(u)u_t(t), A^\varrho y(t)\rangle\right| &\leq C_{\|u(t)\|_{\mathcal{H}^{\frac{7}{4}}}} \|u_t(t)\|_{L^2}\|y(t)\|_{\mathcal{H}^{\varrho+1}}, \\ 2\left|\langle f_0'(z)z_t(t), A^\varrho y(t)\rangle\right| &\leq C_{\|z(t)\|_{\mathcal{H}^{\frac{7}{4}}}} \|z_t(t)\|_{L^2}\|y(t)\|_{\mathcal{H}^{\varrho+1}}. \end{aligned} \tag{3.66}$$

Applying Young's inequality, we can absorb the resulting terms into the left-hand side, leading to:

$$\frac{d}{dt}\mathcal{E}_\varrho(t) + \frac{\epsilon}{2}\mathcal{E}_\varrho(t) \le C\left(\|u(t)\|^2_{\mathcal{H}^{\frac{7}{4}}} + \|z(t)\|^2_{\mathcal{H}^{\frac{7}{4}}} + \|u_t(t)\|^2_{L^2} + \|z_t(t)\|^2_{L^2}\right).$$

From previous lemmas (3.1.7), (3.1.9) and (3.1.10), the quantities on the right-hand side are uniformly bounded for $u_0 \in B_\nu \subset \mathcal{H}^{\frac{3}{4}}$. Therefore, there exists a constant $C_0 > 0$ such that:

$$\frac{d}{dt}\mathcal{E}_\varrho(t) + \frac{\epsilon}{2}\mathcal{E}_\varrho(t) \le C_0.$$

Applying Gronwall's inequality gives:

$$\mathcal{E}_\varrho(t) \le \mathcal{E}_\varrho(0)e^{-\frac{\epsilon}{2}t} + \frac{2C_0}{\epsilon} \le C_{\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}}, \quad \forall t \ge 0.$$

Finally, since $\mathcal{E}_\varrho(t)$ controls $\|Y(t,0)u_0\|^2_{\mathcal{H}^\varrho}$, we conclude:

$$\|Y(t,0)u_0\|^2_{\mathcal{H}^\varrho} \le C_{\|B_\nu\|_{\mathcal{H}^{\frac{3}{4}}}}, \quad \forall t \ge 0.$$

□

**Lemme 3.1.13.** *Let $\frac{3}{4} \le \varrho \le 1$, assume the initial data set $B_\varrho$ is bounded, then there exists a positive constant $J_{\|B_\varrho\|_{\mathcal{H}^\varrho}}$ such that*

$$\|U(t,0)u_0\|^2_{\mathcal{H}^\varrho} \le J_{\|B_\varrho\|_{\mathcal{H}^\varrho}} \quad \textit{for all } t \ge 0 \textit{ and } u_0 = (u(0), u_t(0)) \in B_\varrho.$$

*Proof.* Consider the solution $u(t)$ corresponding to the initial data $u_0 = (u(0), u_t(0)) \in B_\varrho$, with norm

$$\|u_0\|^2_{\mathcal{H}^\varrho} = \|u(0)\|^2_{\mathcal{H}^{\varrho+1}} + \|u_t(0)\|^2_{\mathcal{H}^\varrho}.$$

For a sufficiently small $\epsilon > 0$, define

$$\tilde{u}(t) = u_t(t) + \epsilon u(t).$$

Define the modified energy

$$\mathcal{E}_\varrho(t) = \|\tilde{u}(t)\|^2_{\mathcal{H}^\varrho} + \|u(t)\|^2_{\mathcal{H}^{\varrho+1}} + 2\langle F(u(t)), A^\varrho u(t)\rangle,$$

Under the assumptions on $f$ and coercivity, there exist constants $c_1, c_2 > 0$ such that

$$c_1\|U(t,0)u_0\|^2_{\mathcal{H}^\varrho} \le \mathcal{E}_\varrho(t) \le c_2\|U(t,0)u_0\|^2_{\mathcal{H}^\varrho}.$$

Differentiating $\mathcal{E}_\varrho(t)$ and using the equation satisfied by $u$, we obtain an inequality of the form

$$\frac{d}{dt}\mathcal{E}_\varrho(t) + \epsilon\mathcal{E}_\varrho(t) \le 2\langle f'(u)u_t, A^\varrho u\rangle + 2\langle g(u_t), A^\varrho u\rangle,$$

where $g(u_t)$ represents damping term.

Thanks to Sobolev embeddings and the boundedness of $\|u(t)\|_{\mathcal{H}^{\varrho+1}}$ in a sufficiently high order space, we have

$$|\langle f'(u)u_t, A^\varrho u\rangle| \le C_{\|u\|_{\mathcal{H}^{\varrho+1}}}\|u_t\|_{\mathcal{H}^\varrho}\|u\|_{\mathcal{H}^{\varrho+1}},$$

and similarly,

$$|\langle g(u_t), A^{\varrho}u\rangle| \leq C\|g(u_t)\|_{\mathcal{H}^{\varrho}}\|u\|_{\mathcal{H}^{\varrho+1}}.$$

For any $\delta > 0$, applying Young's inequality, we get

$$2|\langle f'(u)u_t, A^{\varrho}u\rangle| \leq \frac{C^2_{\|u\|_{\mathcal{H}^{\varrho+1}}}}{2\delta}\|u_t\|^2_{\mathcal{H}^{\varrho}} + \frac{\delta}{2}\|u\|^2_{\mathcal{H}^{\varrho+1}},$$

and similarly for the terms involving $g(u_t)$.

Choosing $\delta > 0$ small enough, we obtain :

$$\frac{d}{dt}\mathcal{E}_{\varrho}(t) + \frac{\epsilon}{2}\mathcal{E}_{\varrho}(t) \leq C_0,$$

where the constant $C_0 > 0$ depends only on the bound of $B_{\varrho}$.

By integrating and applying Gronwall's inequality,

$$\mathcal{E}_{\varrho}(t) \leq \mathcal{E}_{\varrho}(0)e^{-\frac{\epsilon}{2}t} + \frac{2C_0}{\epsilon} \leq J_{\|B_{\varrho}\|_{\mathcal{H}^{\varrho}}},$$

with

$$J_{\|B_{\varrho}\|_{\mathcal{H}^{\varrho}}} = \mathcal{E}_{\varrho}(0) + \frac{2C_0}{\epsilon}.$$

By equivalence of norms, this implies

$$\|U(t,0)u_0\|^2_{\mathcal{H}^{\varrho}} \leq \frac{1}{c_1}\mathcal{E}_{\varrho}(t) \leq \frac{J_{\|B_{\varrho}\|_{\mathcal{H}^{\varrho}}}}{c_1},$$

which completes the proof. □

**Lemme 3.1.14.** *([49]) Let $(\mathcal{M}, d)$ be an abstract metric space and let $U(t,0)$ be a Lipschitz continuous dynamical process in $\mathcal{M}$, i.e*

$$d(U(t,0)m_1, U(t,0)m_2) \leq Ce^{Kt}d(m_1, m_2),$$

*for appropriate constants $C$ and $K$ which are independent of $m_i$, $t$. We further assume that there exist three subsets $M_1, M_2, M_3 \subset \mathcal{M}$ such that*

$$\begin{cases} \operatorname{dist}_{\mathcal{M}}(U(t,0)M_1, M_2) \leq C_1e^{-\alpha_1 t}, \\ \operatorname{dist}_{\mathcal{M}}(U(t,0)M_2, M_3) \leq C_2e^{-\alpha_2 t}. \end{cases}$$

*Then*

$$\operatorname{dist}_{\mathcal{M}}(U(t,0)M_1, M_3) \leq C'e^{-\alpha' t},$$

*where $C' = CC_1 + C_2$, $\alpha' = \frac{\alpha_1\alpha_2}{K+\alpha_1+\alpha_2}$.*

### 3.1.4 Proof of Theorem 3.1.1

*Proof.* We are now ready to prove our main results of this section, and the proof based on the a priori estimates in subsection 3.1.2 and 3.1.3 and attraction transitivity result of [25]

From Theorem 3.0.1, we have

$$\operatorname{dist}_{\mathbb{E}}(S(t)\mathcal{B}, \mathfrak{B}_0) \leq \mathcal{K}_0e^{\alpha_1 T_{\mathcal{B}}}e^{-\alpha_1 t}, \tag{3.67}$$

where $\alpha_1 > 0$ and $\mathfrak{B}_0 = \widehat{\mathcal{B}} \times \mathbb{T}^n \subset \mathbb{E}$, and $\mathfrak{B}_\zeta = \mathcal{M}_\zeta \times \mathbb{T}^n \subset \mathbb{E}^1$, with $\mathcal{M}_\zeta \subset \mathcal{H}^{1/4}$ a bounded set.

Let $u(t)$ denote the solution corresponding to initial data $u_0 \in \widehat{\mathcal{B}} \subset \mathcal{H}$.
Using Lemma 3.1.6, the solution admits a decomposition

$$Z(t,0)u_0 = v(t) + w(t),$$

where

- $v(t) \in \mathcal{H}$ decays exponentially:

$$\|v(t)\|_{\mathcal{H}} \leq \mathcal{Q}_1(\|u_0\|_{\mathcal{H}})e^{-\alpha_3 t},$$

- $w(t) \in \mathcal{H}^{1/4}$ is uniformly bounded:

$$\sup_{t\geq 0} \|w(t)\|_{\mathcal{H}^{1/4}} \leq C(\|u_0\|_{\mathcal{H}}).$$

Let us define the following set of regular components:

$$\mathcal{M}_\zeta = \{w(t) \mid u_0 \in \widehat{\mathcal{B}},\ t \geq 0\} \subset \mathcal{H}^{1/4},$$

which is bounded in $\mathcal{H}^{1/4}$ due to the uniform bound above. Then define

$$\mathfrak{B}_\zeta = \mathcal{M}_\zeta \times \mathbb{T}^n \subset \mathbb{E}^1.$$

We now estimate the distance between the full trajectories and the regular set:

$$\operatorname{dist}_{\mathbb{E}}(S(t)\mathfrak{B}_0, \mathfrak{B}_\zeta) = \sup_{u_0\in\widehat{\mathcal{B}}} \inf_{z\in\mathcal{M}_\zeta} \|v(t) + w(t) - z\|_{\mathcal{H}}.$$

Since $w(t) \in \mathcal{M}_\zeta$, we can choose $z = w(t)$, yielding:

$$\|v(t) + w(t) - w(t)\|_{\mathcal{H}} = \|v(t)\|_{\mathcal{H}} \leq \mathcal{Q}_1(\|u_0\|_{\mathcal{H}})e^{-\alpha_3 t}.$$

Therefore, we conclude:

$$\operatorname{dist}_{\mathbb{E}}(S(t)\mathfrak{B}_\zeta, \mathfrak{B}_y) \leq \mathcal{Q}_1(\|\mathcal{M}_v\|_{\mathcal{H}})e^{-\alpha_3 t},$$

where $\mathfrak{B}_v = \mathcal{M}_v \times \mathbb{T}^n$, with $\mathcal{M}_v \subset \mathcal{H}^{3/4}$, and $\mathfrak{B}_\varrho = \mathcal{M}_\varrho \times \mathbb{T}^n$, with $\mathcal{M}_\varrho \subset \mathcal{H}^\varrho$, $\varrho \in \left[\frac{3}{4}, 1\right]$.
We proceed by decomposing the evolution of the semigroup $S(t)\mathfrak{B}_v$ as

$$S(t)\mathfrak{B}_v = v(t) + y(t),$$

where:

- $v(t)$ decays exponentially in $\mathcal{H}$ as a consequence of Lemma 3.1.3:

$$\|v(t)\|_{\mathcal{H}} \leq \mathcal{Q}_1\left(\|u_0\|_{\mathcal{H}}\right) e^{-k_0 t};$$

- $y(t)$ belongs to a bounded subset of $\mathcal{H}^\varrho$ due to Lemma 3.1.12:

$$\|y(t)\|_{\mathcal{H}^\varrho} \leq C, \quad \forall t \geq 0.$$

Now define the attracting set

$$\mathfrak{B}_\varrho = \mathcal{M}_\varrho \times \mathbb{T}^n,$$

where $\mathcal{M}_\varrho = \{y(t) \mid u_0 \in \mathcal{M}_v, t \geq 0\}$, which is bounded in $\mathcal{H}^\varrho$ by Lemma 3.1.12.

Since $y(t) \in \mathcal{M}_\varrho$, for each $u_0 \in \mathcal{M}_v$ we choose $z = y(t) \in \mathfrak{B}_\varrho$, and thus

$$\mathrm{dist}_{\mathbb{E}}(S(t)u_0, \mathfrak{B}_\varrho) \leq \|v(t)\|_{\mathcal{H}} \leq \mathcal{Q}_1(\|u_0\|_{\mathcal{H}})e^{-\alpha_4 t}.$$

Taking $u_0 \in \mathcal{M}_v$, we obtain:

$$\mathrm{dist}_{\mathbb{E}}(S(t)\mathfrak{B}_v, \mathfrak{B}_\varrho) \leq \mathcal{Q}_1(\|\mathcal{M}_\varrho\|_{\mathcal{H}})e^{-\alpha_4 t}, \tag{3.68}$$

where $\mathfrak{B}_\varrho = \mathcal{M}_\varrho \times \mathbb{T}^n$, $\mathcal{M}_\varrho$ is bounded in $\mathcal{H}^\varrho$ and $\alpha_4$ depends on the $\mathcal{H}$-bounds of $\mathcal{M}_\varrho$.

Combining now (3.63)-(3.64), recalling Lemma 3.0.3, and applying Lemma 3.1.13, we deduce

$$\mathrm{dist}_{\mathbb{E}}(S(t)\mathcal{B}, \mathcal{M} \times \mathbb{T}^n) \leq \mathcal{Q}(\|\mathcal{B}\|_{\mathbb{E}})e^{-\alpha t}, \quad \text{for all } t \geq 0,$$

where $\mathcal{M} = \mathcal{M}_\varrho$ and $\alpha$ only depends on $\alpha_i$ $(i = 0, 1, 2, 3, 4)$. □

**Corollaire 3.1.15.** *Let the assumptions of Theorem 3.1.1 hold. Then the solution semigroup* $\{S(t)\}_{t\geq 0}$ *associated with problem (1.1)-(1.2) possesses a global attractor* $\mathcal{A}$ *in the extended phase space* $\mathbb{E}$. *Moreover, the attractor* $\mathcal{A}$ *is bounded.*

*Proof.* The exponential attraction estimate above ensures asymptotic compactness of the semigroup. Indeed, given any bounded sequence $\{u_n\} \subset \mathbb{E}$ and sequence of times $t_n \to \infty$, the trajectories $S(t_n)u_n$ are eventually exponentially close to the compact set $\mathcal{M} \times \mathbb{R}^n \subset \mathbb{E}^1$, which is compactly embedded into $\mathbb{E}$. Therefore, up to a subsequence, $S(t_n)u_n$ converges in $\mathbb{E}$.

Since the semigroup $\{S(t)\}_{t\geq 0}$ is continuous, has a bounded absorbing set, and is asymptotically compact in $\mathbb{E}$, it follows by standard attractor theory (see[50]) that there exists a unique global attractor $\mathcal{A} \subset \mathbb{E}$, which is compact, invariant, and attracts all bounded subsets of $\mathbb{E}$.

Since $\mathcal{A} \subset \omega(\mathcal{M} \times \mathbb{R}^n)$ , and this set is bounded in $\mathbb{E}^1$, it follows that $\mathcal{A} \subset \mathbb{E}^1$ and is therefore more regular than the original phase space.

We conclude that the semigroup $\{S(t)\}_{t\geq 0}$ possesses a global attractor $\mathcal{A} \subset \mathbb{E}$, which is bounded in $\mathbb{E}^1$. □

## 3.2 Uniform exponential attractor

**Théorème 3.2.1.** *Let the assumptions of Theorem 3.1.1 be satisfied, and assume further that* $f \in C^2(\mathbb{R})$ . *For all* $0 < \theta < \frac{1}{2}$, *we define* $t^* = \frac{1}{\lambda} \ln \frac{1}{\theta}$, *where* $\lambda = \frac{\alpha}{2}$.
*Then, for all* $\kappa \in (0, \frac{1}{2} - \theta)$, *the semigroup* $\{S(t)\}_{t\geq 0}$ *induced by solutions of (1.1)-(1.2) on* $\mathbb{E}$ *has a time-dependent exponential attractor* $\mathbf{M} = \{\mathbf{M}^\kappa(s) \mid s \in \mathbb{R}\}$ *in* $\mathbb{E}^1$ *which satisfies the following properties:*
*i. There exists* $\varpi > 0$ *such that* $\mathbf{M}^\kappa(s) = \mathbf{M}^\kappa(\varpi + s), \quad \forall s \in \mathbb{R}$.
*ii. The family is positive semi-variant, that is*

$$S(t)\mathbf{M}^\kappa(s) \subset \mathbf{M}^\kappa(t + s), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}.$$

*iii. There exists a positive constant* $\beta$ *such that, for every bounded subset* $\mathcal{B}$ *of* $\mathbb{E}$,

$$\sup_{s\in[0,\varpi]} \mathrm{dist}_{\mathbb{E}}(S(t)\mathcal{B}, \mathbf{M}^\kappa(s)) \leq \mathcal{Q}(\|\mathcal{B}\|_{\mathbb{E}})e^{-\beta t}, \quad \forall t \geq 0.$$

*iv.* $\mathbf{M}^{\kappa}(s)$ *is compact in* $\mathbb{E}^1$ *and its fractal dimension in* $\mathbb{E}^1$ *is uniformly bounded, i.e.,*

$$\sup_{s\in\mathbb{R}} \dim_{\mathfrak{F}}^{\mathbb{E}^1}(\mathbf{M}^{\kappa}(s)) \leq \log \frac{1}{2(\kappa+\theta)} \left( N^{nz}_{\frac{k}{l^*}}(B_1^z(0)) \right), \quad \textit{for all } s \in \mathbb{R}.$$

*where* $Z$, $nZ$ *and* $l^*$ *will be difined later*

*Proof.* Using Lemma 3.1.13, there exists a bounded set $\mathfrak{B}_{\mathcal{S}} \subset \mathbb{E}^1$, such that for all bounded set $\mathcal{B} \subset \mathbb{E}^1$, the following estimates hold:

$$\operatorname{dist}_{\mathbb{E}}(S(t)\mathcal{B}, \mathfrak{B}_{\mathcal{S}}) \leq \mathcal{K}_1 e^{\alpha_5 T_B} e^{-\alpha_5 t},$$

where $\alpha_5 > 0$ and $\mathcal{K}_1 = \sup\{\|S(t)\mathcal{B}\|_{\mathbb{E}^1}, 0 \leq t \leq T_B\} < \infty$. Assume that

$$S(t)\mathfrak{B}_{\mathcal{S}} \subset \mathfrak{B}_{\mathcal{S}}. \tag{3.69}$$

Now, we show the Lipschitz continuity and quasi-stability of the semigroup $S(t)$ in the extended space $\mathbb{E}^1 = \mathcal{H}^1 \times \mathbb{T}^n$. Obviously, $w = u^1 - u^2$ solves

$$w_{tt} + \ell_1(t) w_t + \ell_2(t)\big(\|u_t^1\|^2 - \|u_t^2\|^2\big)(u_t^1 + u_t^2) + Aw + f(u^1) - f(u^2) = 0,$$

where

$$\begin{aligned} \ell_1(t) &= \frac{1}{2}\Big(|u_t^1|^{q-1}) + |u_t^2|^{q-1})\Big) \geq |u_0|^{q-1}, \\ \ell_2(t) &= \frac{1}{2}\int_0^1 \Big(s|u_t^1|^{q-1} + (1-s)|u_t^2|^{q-1}\Big)' ds \geq 0. \end{aligned} \tag{3.70}$$

Using the multiplier $A\tilde{w} = A(w_t + \epsilon w)$ and using (3.70), we have

$$\frac{d}{dt}\Big(\|A^{\frac{1}{2}}\tilde{w}(t)\|^2 + \epsilon\|Aw(t)\|^2\Big) + 2(\alpha-\epsilon)\|A^{\frac{1}{2}}\tilde{w}(t)\|^2 + 2\epsilon\|Aw(t)\|^2 = \sum_{i=1}^{4}\mathcal{J}_i, \tag{3.71}$$

where

$$\begin{aligned} \mathcal{J}_1 &= 2\epsilon\left(\ell_1(t) - \epsilon\right)\langle A^{\frac{1}{2}}w(t), A^{\frac{1}{2}}\tilde{w}(t)\rangle, \\ \mathcal{J}_2 &= -2\ell_2(t)(\|u_t^1(t)\|^2 - \|u_t^2\|^2)\langle A^{\frac{1}{2}}u_t^1(t) + u_t^2(t), A^{\frac{1}{2}}\tilde{w}(t)\rangle, \\ \mathcal{J}_3 &= -2\langle f(u^1(t)) - f(u^2(t)), A\tilde{w}(t)\rangle, \\ \mathcal{J}_4 &= 2\langle 0, A\tilde{w}(t)\rangle. \end{aligned}$$

We now estimate $\mathcal{J}_i$ $(i = 1, 2, 3, 4)$ one by one. Using Young's inequality, we obtain

$$\begin{aligned} |\mathcal{J}_1| = \left|2\epsilon(\ell_1(t)-\epsilon)\langle A^{\frac{1}{2}}w(t), A^{\frac{1}{2}}\tilde{w}(t)\rangle\right| &\leq 2\epsilon(\beta-\epsilon)\left|\langle A^{\frac{1}{2}}w(t), A^{\frac{1}{2}}\tilde{w}(t)\rangle\right|. \\ &\leq \frac{1}{8}\alpha\|A^{\frac{1}{2}}\tilde{w}(t)\|^2 + C_{\epsilon,\alpha,\beta}\|A^{\frac{1}{2}}w(t)\|^2. \end{aligned} \tag{3.72}$$

Using Lemma 3.1.13 and Cauchy-Schwarz inequality, and Young's inequality, we deduce

$$
\begin{aligned}
|\mathcal{J}_2| &= \left|2\ell_2(t)(\|u_t^1(t)\|^2 - \|u_t^2(t)\|^2)\langle A^{\frac{1}{2}}(u_t^1 + u_t^2), A^{\frac{1}{2}}\tilde{w}(t)\rangle\right| \\
&\leq (|u_{Max}|^{q-1})'\|u_t^1(t) + u_t^2(t)\| \cdot \|u_t^1(t) - u_t^2(t)\| \cdot \|A^{\frac{1}{2}}(u_t^1(t) + u_t^2(t))\| \cdot \|A^{\frac{1}{2}}\tilde{w}(t)\| \\
&\leq C_{M,(|u_{Max}|^{q-1})',J_{\|B_1\|_{\mathbb{X}^1}}}\|u_t^1(t) - u_t^2(t)\| \cdot \|A^{\frac{1}{2}}\tilde{w}(t)\| \\
&\leq \frac{1}{8}\alpha\|A^{\frac{1}{2}}\tilde{w}(t)\|^2 + C_{|u_0|^{q-1},M,(|u_{Max}|^{q-1})',J_{\|\Pi_{\mathbb{X}}\mathfrak{B}_S\|_{\mathcal{H}^1}}}\|\partial_t w(t)\|^2,
\end{aligned}
$$

where $(|u_{Max}|^{q-1})' = \max\limits_{r\in[0,M]}|(|u(r)|^{q-1})'|$ and $\Pi_{\mathcal{H}}$ denotes the bounded projection from $\mathbb{E}$ to $\mathcal{H}$.

$$
\begin{aligned}
|\mathcal{J}_3| &= 2\left|\langle f'(u^1(t))\nabla u^1(t) - f'(u^2(t))\nabla u^2(t), A^{\frac{1}{2}}\tilde{w}(t)\rangle\right| \\
&= \left|\langle \ell_3(t)A^{\frac{1}{2}}w(t), A^{\frac{1}{2}}\tilde{w}(t)\rangle\right| + \left|\langle \ell_4(t)w(t)A^{\frac{1}{2}}(u^1(t) + u^2(t)), A^{\frac{1}{2}}\tilde{w}(t)\rangle\right|,
\end{aligned}
$$

where $\ell_3(t) = f'(u^1(t)) + f'(u^2(t))$ and $\ell_4(t) = \int_0^1 f''(su^1(t) + (1-s)u^2(t))ds$.
By Assumption (3.1)-(3.4) and Lemma 3.1.13, and using $\mathcal{H}^2 \hookrightarrow L^\infty$, we get

$$
|\mathcal{J}_3| \leq \frac{1}{8}\alpha\|A^{\frac{1}{2}}\tilde{w}(t)\|^2 + C_{q,\alpha,J_{\|\Pi_{\mathcal{H}}\mathfrak{B}_S\|_{\mathcal{H}^1}}}(\|A^{\frac{1}{2}}w(t)\|^2 + \|w(t)\|^2).
$$

And, we have

$$
|\mathcal{J}_4| \leq \frac{1}{8}\alpha\|A^{\frac{1}{2}}\tilde{w}(t)\|^2 + C_\alpha\|\omega_1 - \omega_2\|^2_{\mathbb{T}^n}. \tag{3.73}
$$

Now substituting (3.72)-(3.73) into (3.71) and choosing $\epsilon$ small enough, we find:

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\tilde{w}(t)\|^2_{\mathcal{H}^1} + \|w(t)\|^2_{\mathcal{H}^2}\right) + \lambda\left(\|\tilde{w}(t)\|^2_{\mathcal{H}^1} + \|w(t)\|^2_{\mathcal{H}^2}\right) \\
&\qquad \leq \mathcal{K}_2\left(\|w(t)\|^2_{\mathcal{H}^1} + \|w(t)\|^2 + \|\omega_1 - \omega_2\|^2_{\mathbb{T}^n}\right),
\end{aligned} \tag{3.74}
$$

where $\lambda = \frac{\alpha}{2}$ and $\mathcal{K}_2$ depend on $q, \epsilon, \alpha, \beta, (|u_{Max}|^{q-1})'$ and $J_{\|\Pi_{\mathcal{H}}\mathfrak{B}_S\|_{\mathcal{H}^1}}$.
Applying Gronwall's inequality, we obtain

$$
\|S(t)(u^1, \omega_1) - S(t)(u^2, \omega_2)\|_{\mathbb{E}_1} \leq e^{\alpha_6 t}\|(u^1, \omega_1) - (u^2, \omega_2)\|_{\mathbb{E}^1}, \tag{3.75}
$$

for all $t \geq 0$ and $(u^i) \in \mathfrak{B}_S (i = 1, 2)$, and $\alpha_6$ depend on $\mathcal{K}_2$ and $\lambda_1$.
So the Lipschitz continuity holds.

Define the space

$$
\mathbb{Z} = \mathcal{W}_T \times \mathbb{T}^n, \tag{3.76}
$$

where

$$
\mathcal{W}_T = \{(w, w_t) : \|(w, w_t)\|^2_{\mathcal{W}_T} = \int_0^T (\|w(t)\|^2_{\mathcal{H}^2} + \|w_t(t)\|^2_{\mathcal{H}^1})dt < \infty\},
$$

with an appropriate $T$.
The norm in $\mathbb{Z}$ is given by

$$
\|Z\|^2_{\mathbb{Z}} = \|(w, w_t)\|^2_{\mathcal{W}_T} + \|\omega\|^2_{\mathbb{T}^n}, \quad Z = (w, w_t, \omega) \in \mathbb{Z}.
$$

Let

$$\mathbf{n}^2_{\mathbb{Z}}(Z) = \int_0^T \|w(t)\|^2_{\mathcal{H}^1} dt + \|\omega\|^2_{\mathbb{T}^n}, \quad Z = (w, w_t, \omega) \in \mathbb{Z}. \tag{3.77}$$

Obviously, $\mathbf{n}_{\mathbb{Z}}(\cdot)$ is a compact seminorm on $\mathbb{Z}$.

Now, we can apply to (3.74) the Gronwall's inequality, and deduce

$$\begin{aligned}\|S(t)(u^1, \omega_1) - S(t)(u^2, \omega_2)\|_{\mathbb{E}^1} &\leq 4e^{-\lambda t}\|(u^1(0), u^1_t(0), \omega_1) - (u^2(0), u^2_t(0), \omega_2)\|_{\mathbb{E}^1} \\ &+2\mathcal{K}_2 \int_0^t e^{-\lambda(t-r)}\big((1+\lambda_1^{-1})\|w(r)\|^2_{\mathcal{H}^1} + \|\omega_1 - \omega_2\|^2_{\mathbb{T}^n}\big) dr. \end{aligned} \tag{3.78}$$

Taking $T$ large enough such that $0 < \theta = 4e^{-\lambda T} < \frac{1}{2}$, we infer from estimate (3.78) that

$$\begin{aligned}\|S(T)(u^1, \omega_1) - S(T)(u^2, \omega_2)\|_{\mathbb{E}^1} &\leq \theta\|(u^1(0), u^1_t(0), \omega_1) - (u^2(0), u^2_t(0), \omega_2)\|_{\mathbb{E}^1} \\ &+ \mathcal{K}_3 \cdot \mathbf{n}_{\mathbb{Z}}(S(t)(u^1, \omega_1) - S(t)(u^2, \omega_2)),\end{aligned} \tag{3.79}$$

for some positive constant $\mathcal{K}_3$ only depend on $\mathcal{K}_2$, $T$, $\lambda$ and $\lambda_1$.

By (3.75), we deduce

$$\begin{aligned}\|(S(t)(u^1, \omega_1) - S(t)(u^2, \omega_2))\|^2_{\mathbb{Z}} &= \int_0^T \big(\|w(t)\|^2_{\mathcal{H}^2} + \|w_t(t)\|^2_{\mathcal{H}^1}\big) dt + \|\omega\|^2_{\mathbb{T}^n} \\ &\leq \ell^* \|(u^1, \omega_1) - (u^2, \omega_2)\|^2_{\mathbb{E}^1},\end{aligned} \tag{4.80}$$

where $\ell^* = \frac{e^{2\lambda_6 \alpha_6 T}}{2\alpha_6}$.

Therefore, combining (3.69), (3.75), (3.79) and (3.80), then using Theorem 1.4.7, we conclude that the semigroup $\{S(t)\}_{t\geq 0}$ has a time-dependent exponential attractor $\mathbf{M} = \{\mathbf{M}(s)\}_{s\in\mathbb{R}}$ in $\mathbb{E}_1$, satisfying (i), (ii) and (iv). Moreover, there exists a positive constant $\alpha_7$, such that for any bounded in $\mathbb{E}^1$ set $\mathcal{B}^1 \subset \mathbb{E}^1$, it holds

$$\sup_{s\in[0,\varpi]} \operatorname{dist}_{\mathbb{E}^1}(S(t)\mathcal{B}^1, \mathbf{M}(s)) \leq Ce^{-\alpha_7 t}, \quad \forall t \geq 0.$$

Using Lemma 3.2.14, we have

$$\sup_{s\in[0,\varpi]} \operatorname{dist}_{\mathbb{E}}(S(t)\mathcal{B}, \mathbf{M}(s)) \leq \mathcal{Q}(\|\mathcal{B}\|_{\mathbb{E}})e^{-\alpha t}, \quad \forall t \geq 0,$$

for any bounded set $\mathcal{B} \subset \mathbb{E}$, and $\beta = \frac{\alpha\alpha_7}{\alpha+\alpha_6+\alpha_7}$. □

# Bibliography

[1] Ahmed Y. Abdallah: Uniform exponential attractors for first order non-autonomous lattice dynamical systems. *J. Differential Equations*, 251 (2011), 1489–1504.

[2] A. Babin, B. Nicolaenko: Exponential attractors of reaction-diffusion systems in an unbounded domain. *J. Dyn. Diff. Eqns.*, 7 (1995), 567–590.

[3] T. Caraballo, G. Lukaszewicz, J. Real: Pullback attractors for asymptotically compact non-autonomous dynamical systems. *Nonlinear Anal.*, 64 (2006), 484–498.

[4] A.N. Carvalho, J.A. Langa, J.C. Robinson: Attractors for Infinite-dimensional Non-autonomous Dynamical Systems. *Applied Mathematical Sciences*, vol. 182, Springer, New York, (2013).

[5] A.N. Carvalho, S. Sonner: Pullback exponential attractors for evolution processes in Banach spaces: theoretical results. *Commun. Pure Appl. Anal.*, 12 (2013), 3047-3071.

[6] M.M. Cavalcanti, V.N. Domingos Cavalcanti, M.A. Jorge Silva, C.M. Webler: Exponential stability for the wave equation with degenerate nonlocal weak damping. *Israel J. Math.*, 219 (2017), 189–213.

[7] I. Chueshov: Long-time dynamics of Kirchhoff wave models with strong nonlinear damping. *J. Differential Equations*, 252 (2012), 1229–1262.

[8] I. Chueshov: Dynamics of Quasi-Stable Dissipative Systems. *Springer, New York*, (2015).

[9] I. Chueshov, I. Lasiecka: Long-time Behavior of Second Order Evolution Equations with Nonlinear Damping. *Mem. Amer. Math. Soc*, vol. 195, (2008).

[10] J.K. Hale: Asymptotic Behaviour of Dissipative Dynamical Systems. *Mathematical Surveys and Monographs. 25, Amer. Math. Soc., Providence*, (1988).

[11] A. Haraux.: Recent results on semilinear wave equations with dissipation, in Semigroups, Theory and Applications. *Vol. I, Pitman Res. Notes Math. Ser.* 141, Longman, Harlow, UK, (1986), 150-157.

[12] H.T. Jiang, C.D. Zhao: Trajectory statistical solutions and Liouville type theorem for nonlinear wave equations with polynomial growth. *Advances in Differential Equations*. 3-4, (2021) 107-132.

[13] G. Kirchhoff : Vorlesungen uber mathematische Physik-Mechanik, *3 Edition, Teubner, Leipzig*, (1883).

[14] H. Lange, G. Perla Menzala: Rates of decay of a nonlocal beam equation. *Differential Integral Equations*. 10, (1997) 1075-1092.

[15] L.A. Medeiros, J.L. Ferrel, S.B. de Menezes: Vibration of elastic strings: mathematical aspects, I. *J. Comput. Anal. Appl.* 4, (2002) 91-127.

[16] L.A. Medeiros, J.L. Ferrel, S.B. de Menezes : Vibration of elastic strings: mathematical aspects, II. *J. Comput. Anal. Appl.* 4, (2002) 211-263.

[17] P. Pucci, S. Saldi : Asymptotic stability for nonlinear damped Kirchhoff systems involving the fractional p-Laplacian operator. *J. Differ. Equ.* 263, (2017) 2375-2418.

[18] J.C. Robinson : Infinite-Dimensional Dynamical Systems. *Cambridge University Press, Cambridge*, (2001).

[19] C.Y. Sun, D.M. Cao, J.Q. Duan: Non-autonomous dynamics of wave equations with nonlinear damping and critical nonlinearity. *Nonlinearity*. 19, (2006) 2645-2665.

[20] C.Y. Sun, L. Yang, J.Q. Duan : Asymptotic behavior for a semilinear second order evolution equation. *Trans. Amer. Math. Soc.* 363, (2011) 6085-6109.

[21] C.Y. Sun, M.H. Yang: Dynamics of the nonclassical diffusion equations. *Asymptotic Anal.* 59, (2008) 51-81.

[22] C.Y. Sun, M.H. Yang, C.K. Zhong : Global attractors for the wave equation with nonlinear damping. *J. Differential Equations*. 227, (2006) 427-443.

[23] R. Temam. : Infinite-Dimensional Dynamical Systems in Mechanics and Physics. *Springer-Verlag, New York*, 1997.

[24] M.H. Yang, J.Q. Duan, P.E. Kloeden : Asymptotic behavior of solutions for random wave equations with nonlinear damping and white noise. *Nonlinear Anal. RWA*. 12, (2011) 464-478.

[25] S.V. Zelik : Asymptotic regularity of solutions of a nonautonomous damped wave equation with a critical growth exponent. *Comm. Pure Appl. Anal. 3*, (2004) 921-934.

[26] F. Zhou, C. Sun, X. Li: Dynamics for the damped wave equations on time-dependent domains. *Discrete Contin. Dyn. Syst. Ser. B*, 23 (2018), 1645-1674.

[27] F. Zhou, H.F. Li, K.X. Zhu: Long-time behavior of wave-type equations with nonlocal weak damping and quintic nonlinearity. *Submitted.*

[28] S.F. Zhou, X.Y. Han: Uniform exponential attractors for non-autonomous KGS and Zakharov lattice systems with quasiperiodic external forces. *Nonlinear Anal.* 78, (2013) 141-155.

[29] Feng Zhou , Kaixuan Zhu and Yongqin Xie .: Dynamics of the non-autonomous wave equations with nonlocal weak damping and critical nonlinearity.

[30] Abbes Benaissa and Aissa Benguessoum : Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the non-linear internal feedback. *Int. J. Dynamical Systems and Differential Equations, Vol5, No 1* , (2014).

[31] Benaissa, A. and Louhibi: Global existence and Energy decay of solutions to a nonlinear wave equation with a delay term *Georgian Mathematical Journal, Vol20, No1*,(2013)1-4

[32] Bermini, S. and Messaoudi, S.A: Existence and decay of solutions of a viscoelastic equation with a nonlinear source *Nonlinear Analysis: Theory, Methods & Applications Vol64* (2006)2314-2331.

[33] Kirane, M. and Said-Houari: Existence and asymptotic stability of a viscoelastic wave equation with a delay *Zeitschrift für Angewandte Mathematik und Physik, Vol62,*(2011)1065-1082.

[34] Nicaise, S. and Pignotti, C: Stabilization of the wave equation with boundary or internal distributed delay *Differential and Integral Equations, Vol 1,*(2008)935-958.

[35] Schaefer, H. H., and Wolff, M. P: traduction de Topological Vector Spaces *Analyse fonctionnelle,*(2007)37-47.

[36] W. Rudin: Functional Analysis, 2nd ed., McGraw-Hill, New York, (1991)46-69.

[37] M. A. Efendiev and A. Miranvillem Exponential attractors and finite-dimensional reduction for nonautonomous dynamical systems, *J. Dynam. Differential Equations*, 17(3), (2005), pp. 593–617.

[38] H. Brezis: Operateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert, Notas de Matematica (50), *Universidade Federal do Rio de Janeiro and University of Rochester, North-Holland, Amsterdam,* (1973).

[39] Olver, Peter J: Introduction to partial differential equations. *Vol. 1. Berlin: Springer*, (2014).

[40] W. Walter: Ordinary Differential Equations, *Springer-Verlage, New York, Inc,* (1998).

[41] R. A. Adams and J. J. F. Fournier: Sobolev Spaces, 2nd ed., *Academic Press*, (2003).

[42] H. Brezis: Functional Analysis, Sobolev Spaces and Partial Differential Equations, *Springer*, (2010).

[43] P. R. Halmos: Naive Set Theory, *Springer*, (1974).

[44] J. Simon: Compact sets in the space $L^p(0,T;B)$, *Annali di Matematica Pura ed Applicata*, 146 (1987), 65–96.

[45] M. Renardy and R. C. Rogers: An Introduction to Partial Differential Equations, *Texts in Applied Mathematics, vol. 13, Springer, 2nd ed*, (2004).

[46] A. Pazy: Semigroups of Linear Operators and Applications to Partial Differential Equations, *Applied Mathematical Sciences, vol. 44, Springer-Verlag*, (1983).

[47] C.Y. Sun, M.H. Yang, C.K. Zhong : Exact controllability for the semilinear wave equation. *Journal de Mathématiques Pures et Appliquées.* 4, (1993) 343–366.

[48] Alain Haraux : Attractors for Equations of Mathematical Physics. *Lecture Notes in Mathematics.* 841, (2002) .

[49] Vladimir V. Chepyzhov and Mark I. Vishik: Nonlinear Evolution Equations: Global Behavior of Solutions. *Colloquium Publications.* 49, (1991) .

[50] Roger Temam: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. *Applied Mathematical Sciences.* 68, (1997) .