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Géométrie différentielle

Intitulé

On(ω, c)-periodic and (ω, c)-pseudo periodic functions and applications

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Dedication

I dedicate this work to my beloved parents, whose unwavering support, encouragement, and sacrifices have been the foundation of my journey. To my family and friends, thank you for believing in me during moments of doubt.

This achievement is also dedicated to my professors and mentors, whose guidance and wisdom have shaped my academic path.

Above all, I thank God for granting me strength, patience, and perseverance throughout this endeavor.

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A special thanks to my colleagues and friends, who stood by me through the challenges and shared with me the moments of perseverance and success. Your companionship and motivation made a significant difference.

I pray that this work will be beneficial and that it will contribute, even modestly, to the field of knowledge. May Allah grant us continued success and guide us on the path of righteousness.

Contents

Abstract

The notion of periodicity plays a fundamental role in mathematics. In this This Work we study a new class of functions, which we call (ω, c)-periodic and (ω, c)-pseudo periodic functions. functions. This collection includes periodic, anti-periodic, Bloch and unbounded functions and pseudo periodic, pseudo anti-periodic, pseudo Bloch-periodic, and unbounded functions. The concept of (ω, c)-periodicity, which generalizes classical periodicity by allowing solutions of the form $x(t+\omega) = cx(t)$, We prove that the set conformed by these functions is a Banach space with a suitable norm. Furthermore, we show several properties of this class of functions as the convolution invariance. We present some examples and a composition result. This paper investigates the existence and uniqueness of (ω, c)-periodic a variety of equations with bounded and unbounded linear operators. Using Banach and Schauder fixed point theorems. As an application, we prove the existence and uniqueness of (ω, c)-pseudo periodic mild solutions to the first order abstract Cauchy problem on the real line. Also, we establish some sufficient conditions for the existence of positive (ω, c)-pseudo periodic solutions to the Lasota–Wazewska equation with unbounded oscillating production of red cells.

Résumé

La notion de périodicité joue un rôle fondamental en mathématiques. Dans ce travail, nous étudions une nouvelle classe de fonctions, appelées fonctions (ω, c)-périodiques et (ω, c)-pseudo-périodiques.

Cette collection comprend des fonctions périodiques, antipériodiques, de Bloch et non bornées, ainsi que des fonctions pseudo-périodiques, pseudo-anti-périodiques, pseudo-Bloch-périodiques et non bornées. Le concept de (ω, c)-périodicité, qui généralise la périodicité classique en autorisant des solutions de la forme $x(t + \omega) = cx(t)$,, nous prouve que l'ensemble conformé par ces fonctions est un espace de Banach de norme appropriée. De plus, nous démontrons plusieurs propriétés de cette classe de fonctions, comme l'invariance de convolution. Nous présentons quelques exemples et un résultat de composition. Cet article étudie l'existence et l'unicité de fonctions (ω, c)-périodiques dans diverses équations avec des opérateurs linéaires bornés et non bornés. En utilisant les théorèmes du point fixe de Banach et Schauder, nous démontrons l'existence et l'unicité de solutions douces (ω, c)-pseudo-périodiques du problème abstrait de Cauchy du premier ordre sur la droite réelle. Nous établissons également des conditions suffisantes pour l'existence de solutions positives (ω, c)-pseudo-périodiques de l'équation de Lasota-Wazewska avec production oscillante illimitée de globules rouges.

Introduction

One of the most attractive topics in qualitative theory is the study of the existence of periodic-type solutions to differential equations; this is due to the mathematics interest of qualitative theory and applications in many scientific fields, such as physics, biology and control theory. Consider the following second order linear ordinary differential equation which is a simple form of the Hill's equation

$$y''(t) + \Big[a - 2q\cos(2t)\Big]y(t) = 0, \tag{1}$$

Where $\omega \neq 0$ is a constant. It is known from the Floquet theorem that there exists at least one constant $c \neq 0$ and one nontrivial solution $y(t)$ of Eq. (1) such that

$$y(t+\omega) = cy(t).$$

Let $q(t) := a - 2q\cos(2t)$ in Eq. (1), we then obtain the **Mathieu equation**:

$$y''(t) + \Big[a - 2q\cos(2t)\Big]y(t) = 0,$$

which describes... a linearized model of an inverted pendulum with the pivot point oscillating periodically in the vertical direction and has also many modelling examples in fluid dynamics and seasonally forced population dynamics [45]. It is seen that the property of $y(t+\omega) = cy(t)$ reveals some well-known recurrence Recently, Alvarez et al. [45] introduced the concept of (ω, c)-periodic functions, which encompasses the classes of periodic, antiperiodic, and Bloch periodic functions, among others. This concept is motivated by Mathieu's equation: such as the periodicity ($c = 1$), antiperiodicity ($c = -1$, see [25, 26]) and Bloch periodicity ($c = e^{ik\omega}$, see [16, 20]), and thus $y(t)$ is usually called an$(\omega; c)$-periodic function [31]. Alvarez, G´omez and Pinto [5] originally studied the completeness, convolution and composition theorems for $(\omega; c)$-periodic functions in abstract spaces. The discrete $(\omega; c)$-periodic function in abstract spaces was also presented by Alvarez, D´ıaz and Lizama in [3]. Additionally, some important extensions of $(\omega; c)$-periodic functions in abstract spaces have been made to show the effect of small perturbations on $(\omega; c)$-periodic functions. For instance, Alvarez, Castillo and Pinto [2] introduced notions of $(\omega; c)$-asymptotically periodic functions and $(\omega; c)$-pseudo periodic functions in abstract spaces with applications to the first order abstract Cauchy problem and Lasota-Wazewska model with ergodic and unbounded oscillating production of red cells respectively. For more studies on $(\omega; c)$-periodic functions with extensions and applications, we refer to [6, 5, 7] and references cited therein.

In their paper [2], the authors generalized this concept to (ω, c)-pseudo periodic functions, which are functions with ergodic parts. Both papers [45, 2] extend several results from [7, 10, 11, 12].

Our work is based on the [18], it is divided into three chapters:

In first chapter,We have introduced the different branches of periodic functions and their properties, specializing in (ω, c)- periodic function and (ω, c) pseudo periodic function and their propertirs . We have also introduced some concepts like fixed point theorem, Banach space and linear operators which are important for our study.

In second chapter We study the existence and uniqueness of (ω, c)-periodic solutions for semilinear evolution equations in complex Banach spaces.

In Section 2, we consider the case where the linear operator of the evolution equation is bounded. Assuming a non-resonance condition, we derive a Green's function for a nonhomogeneous linear equation with the corresponding boundary value conditions.

Then, in Section 3, we rewrite our problem as a fixed point equation and solve it via the Banach fixed point theorem, obtaining an existence and uniqueness result for (ω, c)-periodic solutions. In Section 4, the Schauder fixed point theorem is applied to prove an existence result for the problem introduced in Section 3.

In Section 5, we extend the considerations of Sections 3 and 4 to the case of evolution equations with unbounded linear operators. All theoretical results are illustrated by several examples.

In third chapter In Section 1. We prove the existence and uniqueness of (ω, c)-pseudo periodic mild solutions to the first order abstract Cauchy problem on the real line. In particular, we obtain (ω, c)-pseudo periodic mild solutions for the semilinear first order problem

$$u'(t) = Au(t) + f(s, u(s))\, ds,\ \ t \in \mathbb{R},$$

where A is a closed linear and densely defined operator on a Banach space X which generates an exponentially bounded C0-semigroup $\{T(t)\}_{t\geq 0}$. We also consider the case where the operator A generates an exponentially stable C_0-semigroup, leading to further existence results for associated semilinear evolution equations.

In Section 2, we establish some sufficient conditions for the existence of positive (ω, c)-pseudo periodic solutions to the Lasota–Wazewska equation with unbounded oscillating production of red cells.

$$y(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)},$$

Wazewska–Czyzewska and Lasota [24] proposed this model to describe the survival of red blood cells in the blood of an animal. In this equation, y(t) describes the number of red cells bloods in the time t, $\delta > 0$ is the probability of death of a red blood cell, a(t) is a continuous and positive function which is related to the production of red blood cells by unity of time, τ is the time required to produce a red blood cell, $h(t)$ is a continuous and positive function which describes the generation of red blood cells per unit time.

Chapter 1

Preliminary

Sommaire

1.1 Banach space

1.1.1 Norms on a vector space

In this chapter, we cover essential concepts, notations, defnitions, theorems, and properties necessary to develop our primary fnding

Definition 1.1.1. *Norm We take V an $\mathbb{R}-vector space.$A Norm is a map defined on V to values in $\mathbb{R}^+$, denoted $\|.\|_V$ and such that the following three properties are satisfied:*
(i) $\forall v \in V, \|v\|_V = 0 \Leftrightarrow v = 0$;
(ii) $\forall \lambda \in \mathbb{R}, \forall v \in V, \|\lambda v\|_V = |\lambda| \|v\|_V$;
(iii) Triangle inequality $\forall v, w \in V, \|v + w\|_V \leq \|v\|_V + \|w\|_V$;
We then say V is an $\mathbb{R}$-normalized vector space.

Example 1.1.2. *On $\mathbb{R}^n$, we can defined the classic norms:*
For $x = (x_1, ..., x_n) \in \mathbb{R}^n$:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|,$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}},$$

$$\|x\|_\infty = \sup_{i=1,..,n} |x_i|.$$

Example 1.1.3. *Let* $[a,b]$ *be an interval in* $\mathbb{R}$*, let us denoted* $C([a,b])$ *the vector space made up of continuous functions on* $[a,b]$ *and with values in* $\mathbb{R}$*. The application* $\|.\|_\infty$ *which* $f \in C([a,b]$ *associates:*

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|,$$

defines a norm on $C([a,b])$*.*
We can also define another norm on $C([a,b])$*:*

$$\|f\|_R = \int_a^b |f(t)|dt.$$

We can therefore have several norms on the same space.

Equivalance of norms

Definition 1.1.4. *Let* $\|.\|_{V1}$ *and* $\|.\|_{V2}$ *two norms.We say that the two norms equivalents if there exists two constants* c_1 *and* c_2 *are positivs such that:*

$$\forall v \in V, c_1\|v\|_{V2} \leq \|v\|_{V1} \leq c_2\|v\|_{V2}.$$

The importance of this definition comes from the fact that if two norms are equivalent, they induce the same topology.

Proposal 1.1.5. *if* $\|.\|_{V_1}$ *and* $\|.\|_{V_2}$ *are two norms equivalent to* V*, we get the equivalence:* $(u_n)_{n\in\mathbb{N}}$ *converges to* l *for* $\|.\|_{V1} \Leftrightarrow (u_n)_{n\in\mathbb{N}}$ *converges to* l *for* $\|.\|_{V2}$*.*

Proposal 1.1.6. *If* V *has a finite dimension, So, all norms are equivalent.*

1.1.2 Complete Space

Let V is a norm vector space to norm $\|.\|_V$.

Cauchy Sequence

Definition 1.1.7. *Let* $(u_n)_{n\in\mathbb{N}}$ *a sequence in* V*. we say that* $(u_n)_{n\in\mathbb{N}}$ *is Cauchy sequence for the norm* $\|.\|_V$ *if:*

$$\forall\epsilon > 0, \exists N \geq 0, \forall n \geq N, \exists p \geq 0, \|u_{n+p} - u_n\|_V \leq \epsilon.$$

Proposal 1.1.8. *Every Cauchy sequence is bounded.*

Proposal 1.1.9. *Every convergent sequence is a Cauchy sequence.*

1.1.3 Banach space

Notation 1.1.10. *It's not necessary that every Cauchy sequance is convergent in any space. And here we get the new definition:* ***Complete spaces****.*

Definition 1.1.11. *(Complete space or Banach space)*
V is $\mathbb{R}$vector space, it's complete for the norm $\|.\|_V$ if every cauchy sequence (of this norm) is convergent. such a space is called also Banach space.

Proposal 1.1.12. *Every vector space in $\mathbb{R}$ normed with finite dimension is complete. This comes from the fact that $\mathbb{R}$ is complete (by construction,admitted).*

Proposal 1.1.13. *$C([a,b])$ equipped with the norm $\|.\|_\infty$ is a Banach space.*
Proof: *Let $(u_n)_{n\geq 0}$ a Cauchy Sequence in $C([a,b])$, i.e.,*

$$\forall \epsilon > 0, \exists N \geq 0, \forall n \geq N, \forall p \geq 0, \|U_{n+p} - u_n\|_\infty = \sup_{x\in[a,b]} |U_{n+p}(x) - u_n(x)| \leq \epsilon.$$

For all $x \in [a,b]$, $(u_n(x)_{n\in\mathbb{N}})$ is a cauchy sequence in $\mathbb{R}$ and convergent to a limit noted $u(x)$. By making p tend towards infinity, we obtain:

$$\forall \epsilon > 0, \exists N \geq 0, \forall n \geq N, \sup_{x\in[a,b]} |u(x) - u_n(x)| \leq \epsilon.$$

We proove that u is a continous function in $[a,b]$. Let $x \in [a,b]$ and $\epsilon > 0$. we say that there exists $N \in \mathbb{N}$ such that:

$$\sup_{z\in[a,b]} |u(z) - u_N(z)| \leq \frac{\epsilon}{3}.$$

Moreover, u_N being continuous, there exists $\eta > 0$ such that for all $y \in [a,b]$ et $\epsilon \succ 0$ such that $|y-x| \leq \eta$, we have

$$|u_n(y) - u_n(x)| \leq \frac{\epsilon}{3}.$$

Triangular inequality allows us to conclude: for all $y \in [x-\eta; x+\eta] \cap [a,b]$, we have:

$$|u(y) - u(x)| \leq |u(y) - u_N(y)| + |u_N(y) - u_N(x)| + |u_N(x) - u(x)| \leq \epsilon.$$

It means that $u \in C([a,b])$. We conclude by noting that $(u_n)_{n\geq 0}$ converges to u in $C([a,b])$
Against example: *the function space $C([a,b])$ provided with the norm $\|.\|_R$ is not complete.*

Remark 1.1.14. *It is easy to show that a sequence is cauchy that it is convergent (we do not need to know the limit of the sequence).*

1.2 Fixed point theorems

The theory of fixed point is one of the most powerful tools of modern mathematics. The theorems which are concerning with the existence of solutions for differential equations.

Definition 1.2.1. *A point $x \in X$ is called a fixed point of an operator $L : X \to X$, if*

$$L(x) = x, x \in X.$$

Banach Contraction Principle

Theorem 1.2.2. *(Banach fixed point theorem)*
Let (X, d) be a nonempty complete metric space, and let $0 \leq \lambda < 1$. If $T : X \to X$ is a mapping such that for every $x, \bar{x} \in X$, the relation

$$d(Tx, T\bar{x}) \leq \lambda d(x, \bar{x})$$

holds, then the operator T has a unique defined fixed point$x^\star \in X$. Moreover, if $T^k (k \in \mathbb{N})$ is the sequence defined by

$$\begin{cases} T^k = TT^{k-1}, k \in \mathbb{N} \\ T^1 = T \end{cases}$$

Then, for any $x_0 \in X$,$[T^k x_0]_{k=1}^{k=\infty}$ converges to the above fixed point $x^\star$.

Definition of Compactness

In Metric Spaces:
A subset $K \subset \mathbb{R}^n$ is called **compact** if it is both:

- **Closed**: it contains all its limit points,
- **Bounded**: there exists $M > 0$ such that $\|x\| < M$ for all $x \in K$.

This is known as the **Heine–Borel theorem**:

$$K \subset \mathbb{R}^n \text{ is compact } \Leftrightarrow K \text{ is closed and bounded.}$$

In Topological Spaces:
A topological space X is said to be **compact** if every open cover has a finite subcover.
That is, for every collection of open sets $\{U_\alpha\}_{\alpha \in A}$ such that

$$X \subset \bigcup_{\alpha \in A} U_\alpha,$$

there exists a finite subcollection $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ such that

$$X \subset \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Definition: Compact Operator

Let X and Y be Banach spaces (or normed vector spaces), and let $A : X \to Y$ be a linear operator.

We say that A is a **compact operator** if for every bounded subset $B \subset X$, the image $A(B) \subset Y$ has compact closure in Y. That is,

$$\overline{A(B)} \text{ is compact in } Y.$$

Equivalently, A is compact if it maps bounded sequences $(x_n) \subset X$ to sequences $(Ax_n) \subset Y$ that have a convergent subsequence.

Definition: Relatively Compact Set

Let X be a topological space, and let $S \subset X$. We say that S is **relatively compact** in X if the closure of S, denoted $\overline{S}$, is compact in X. That is,

$$S \text{ is relatively compact } \Leftrightarrow \overline{S} \text{ is compact.}$$

In metric spaces: A subset $S \subset X$ is relatively compact if every sequence in S has a convergent subsequence whose limit belongs to X.

Definition 1.2.3. *Let A be a linear operator from a Banach space X to another Banach space Y. A is compact if it transforms every bounded set of X into a relatively compact set of Y.*

Definition 1.2.4. *An operator is said to be completely continuous if it is compact and continuous.*

1.2 Bounded and Convex Sets

1. Bounded Set:
Let $K \subset X$, where X is a normed vector space. We say that K is **bounded** if there exists a constant $M > 0$ such that:

$$\|x\| \leq M \quad \text{for all } x \in K.$$

2. Convex Set:
A subset $K \subset X$ is called **convex** if for all $x, y \in K$ and all $\lambda \in [0, 1]$, we have:

$$\lambda x + (1 - \lambda)y \in K.$$

This means the line segment between any two points of K lies entirely within K.

Ascoli-Arzela theorem

Definition 1.2.5. *For $A \subset C(Xf$, A is compact if, and only if, A is closed, bounded, and equicontinuous.*

Schauder Fixed Point Theorem

Theorem 1.2.6. *(Schauder's fixed-point theorem). If K is a closed bounded and convex subset of a Banach space X and $F : K \to K$ is completely continuous, then F has a fixed point in K.*

1.3 Linear operators

1.3.1 Maps and linear operators

Definition 1.3.1. *Let X be a Banach space. A map $A : D(A) \subset X \to X$ is a linear operator if $D(A)$ is an under vector space of X and A is linear.*
$D(A)$ is the domain of A and the image of this operator is noted by $R(A)$.

Example 1.3.2. *Let A the differential operator $\dfrac{d}{dt}$ where:*

$$D(A) = \{f \in C^1(\mathbb{R}, X) : \frac{d}{dt} \in BC(\mathbb{R}, X)\}.$$

We denote by $L(X)$ the set of linear operators from X to X .
$L(X)$ is an algebra, the internal product is the map composition, and, the neutral element is I the identity map.
An element T in $L(X)$ is reversible, if there exists alinear map S such that:
$T \circ S = S \circ T = I$.
Si $T \in L(X)$, the following propreties are equivalents:
i) T is reversible;
ii) T is bijective and and T^{-1} is continued;
iii) $\ker T = 0$ and $ImT = X$ and T^{-1} is continued.

Proposal 1.3.3. *The set of reversibles elements of $L(X)$ is an open of $L(X)$which contains I.*

Definition 1.3.4. *Let A be a linear operator on a Banach space X with $D(A) = X$.*
A is a bounded linear operator in X if there exists a positive constant η such that:

$$\|Ax\| \leq \eta \|x\|, \forall x \in X.$$

Definition 1.3.5. *A linear operator $A : X \to X$ is continuous on $x \in X$ if for every sequence $(x_n) \subset X$ such that $x_n \to x$, we have $Ax_n \to Ax$, which means : $\|Ax_n - Ax\| \to 0$ when $\|x_n - x\| \to 0$.*

Definition 1.3.6. *A linear operator $A : X \to X$ is continuous on X if and only if, it is continued on all point $x \in X$.*

Remark 1.3.7. *A is a linear operator bounded if and only if A is continued.*

Definition 1.3.8. *Let A be a linear operator. The following number is called the norm of A:*

$$\|A\| := \inf\{\eta \in \mathbb{R} : \|A\| \leq \eta \|x\|, \forall x \in X\}.$$

Definition 1.3.9. *A linear operator A defined on $D(A) \subset X$ has a values in X is closed if the graph $\{(x, Ax), x \in D(A)\}$ is closed in $X \times X$.*

1.4 Periodic functions

Definition 1.4.1. *If $f : \mathbb{R} \to X$ is a function, we denote the translate of f by $s \in \mathbb{R}$ the function defined by:*

$$R_s f(t) = f(t+s)$$

$\forall t \in \mathbb{R}$.

Definition 1.4.2. *(Periodic function): A function $f \in C(\mathbb{R}, X)$ is said to be Periodic, if there exists $\tau \in \mathbb{R}$,such that:*

$$R_\tau f(t) = f(t), \forall t \in \mathbb{R}.$$

In this case,τ is called a periodic of the function f.

Example 1.4.3. *Functions such as $f(t) = \frac{1}{3}\sin(2t)$, $h(\theta) = 1-\cos(\theta)$, $g(x) = -2\cos(4x+5)$ are periodic, it is easy to verify that the number τ is exists for each of these functions.*

Example 1.4.4. *let us take the function defined as follows:*

$$f(t) = \sin(\sqrt{2t}) + \sqrt{3}\cos(t)$$

This function, althought continuous, it is not periodic.

Solution of the Problem
Let us consider the function:

$$f(t) = \sin(\sqrt{2t}) + \sqrt{3}\cos(t).$$

Assume, for contradiction, that f is periodic with some period $T > 0$. Then for all $t \in \mathbb{R}$, we would have:

$$f(t+T) = f(t),$$

which implies:

$$\sin(\sqrt{2(t+T)}) + \sqrt{3}\cos(t+T) = \sin(\sqrt{2t}) + \sqrt{3}\cos(t).$$

Since $\cos(t)$ is a periodic function with period 2π, the term $\sqrt{3}\cos(t+T)$ is also periodic. Therefore, it is possible for the cosine term to repeat if T is a suitable multiple of 2π.
However, the function $\sin(\sqrt{2t})$ is not periodic. This is because its argument, $\sqrt{2t}$, is not linear in t, and the difference

$$\sqrt{2(t+T)} - \sqrt{2t}$$

is not constant, nor does it produce values which differ by a multiple of 2π for all t. Hence, $\sin(\sqrt{2t})$ does not repeat regularly, and is not a periodic function.

Example 1.4.5. *in this figure we will see the graphics of* cos *and* sin *which are a* 2π *periodic functions.*
[h]

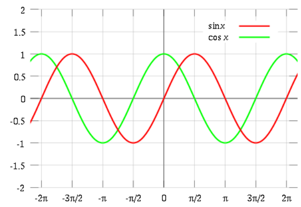

Figure 1.1: Graphs of the periodic function cosine and Sin

1.4 (ω, c)-Periodic Functions and Their Properties

Throughout the paper, $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$, X will denote a complex Banach space with norm $\|\cdot\|$, and the space of continuous functions as

$$C(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is continuous}\}.$$

Definition 1.4.6. *A function $f \in C(\mathbb{R}, X)$ is said to be (ω, c)-periodic if*

$$f(t + \omega) = cf(t) \quad \text{for all } t \in \mathbb{R}.$$

ω is called the c-period of f. The collection of those functions with the same c-period ω will be denoted by $P_{\omega c}(\mathbb{R}, X)$. When $c = 1$ (the ω-periodic case), we write $P_\omega(\mathbb{R}, X)$ in place of $P_{\omega 1}(\mathbb{R}, X)$. Using the principal branch of the complex Logarithm (i.e., the argument in $(-\pi, \pi]$), we define

$$c^{t/\omega} := \exp((t/\omega)\log(c)).$$

Also, we will use the notation

$$c^\wedge(t) := c^{t/\omega} \quad \text{and} \quad |c|^\wedge(t) := |c^\wedge(t)| = e^{(t/\omega)\ln|c|}.$$

The following proposition gives a characterization of the (ω, c)-periodic functions.

Properties of(ω, c)-Periodic Functions

- **Multiplicative Property:**

$$f, g \text{ are } (\omega, c)\text{- and } (\omega, d)\text{-periodic} \Rightarrow fg \text{ is } (\omega, cd)\text{-periodic.}$$

- **Linearity:** If f_1, f_2 are (ω, c)-periodic, then for any scalars $a, b \in \mathbb{C}$,

$$af_1 + bf_2 \text{ is also } (\omega, c)\text{-periodic.}$$

- **Transformation to Classical Periodicity:**

$$u(t) = e^{-\mu t} f(t), \quad \mu = \frac{\ln c}{\omega} \quad \Rightarrow \quad u(t+\omega) = u(t),$$

so u is a classical ω-periodic function.

Definition 1.4.7. *Let $f \in C(\mathbb{R}, X)$. Then f is (ω, c)-periodic if and only if*

$$f(t) = c^{\wedge}(t)u(t), \quad c^{\wedge}(t) = c^{t/\omega}, \quad u \in P_\omega(\mathbb{R}, X). \tag{2.1}$$

Proof It is clear that if $f(t) = c^{\omega}(t)u(t)$ then f is a (ω, c)-periodic function. In order to show the inverse statement, let $f \in P_{\text{ex}}(\mathbb{R}, X)$. If we write $u(t) = c^{-\omega}(t)f(t) = c^{-it/\omega}f(t)$, then we have that

$$u(t+\omega) = u(t), \tag{1.1}$$

hence the function $u(t)$ is an ω-periodic function and $f(t) = c^{\omega}(t)u(t)$.

In view of (2.1), for any $f \in P_{\text{ex}}(\mathbb{R}, X)$ we say that $c^{\omega}(t)u(t)$ is the c-factorization of f.

Remark 2.3. From Proposition 2.2, we can write all $f \in P_{\text{ex}}(\mathbb{R}, X)$ as

$$f(t) = c^{\omega}(t)u(t), \tag{1.2}$$

where $u(t)$ is ω-periodic on $\mathbb{R}$. We will call $u(t)$ the periodic part of f. With this convention, an anti-periodic function f can be written as $f(t) = (-1)^{t/\omega}u(t)$, where its anti-period is ω.

For example, $f(t) = \sin(t)$ can be considered as an anti-periodic function, with $\omega = \pi$. As $\log(-1) = i\pi$, f has the decomposition $f(t) = c^{\omega}(t)u(t)$ where

$$c^{\omega}(t) = (-1)^{t/\omega} = e^{it} = \cos t + i \sin t, \tag{1.3}$$

and

$$u(t) = \sin(t(\cos t - i \sin t)). \tag{1.4}$$

Let $c = e^{2\pi i/k}$ for some natural number $k \geq 2$ and let f be a (ω, c)-periodic function, then f is a periodic function with period $k\omega$ but, in general, can be written as $f(t) = e^{2\pi it/k\omega}u(t)$, where u is a complex periodic function with period ω.

In particular, if $k = 4$, an $(\omega, e^{\pi i/2})$-periodic function f can be at the same time a Bloch wave: $f(t+\omega) = e^{i\pi/2}f(t)$, an anti-periodic function with antiperiod 2ω: $f(t+2\omega) = -f(t)$ and a 4ω-periodic function: $f(t+4\omega) = f(t)$.

Remark 1.4.8. *From Definition 2.1 we can observe that $P_{ex}(\mathbb{R}, X)$ is a translation invariant subspace over $C(\mathbb{R}, X)$. Furthermore, $f \in P_{ex}(\mathbb{R}, X)$ derivable implies that $f' \in P_{ex}(\mathbb{R}, X)$ and if $|c| = 1$ then $P_{ex}(\mathbb{R}, X)$ has only bounded functions. If $|c| < 1$ then any element $f \in P_{ex}(\mathbb{R}, X)$ goes to zero when $t \to \infty$, and f is unbounded when $t \to -\infty$, and if $|c| > 1$ then f is unbounded when $t \to \infty$ and f goes to zero when $t \to -\infty$.*

Example 2.4.1 If we consider the linear delayed equation

$$x'(t) = -px(t - r), \quad t \in \mathbb{R}, \tag{1.5}$$

with $p, r > 0$, a solution $\phi(t) = e^{st}$, with $z0 = x0 = \phi(0) = y0, y0 \in \mathbb{R}, y0 > 0$, where $s0 + pe^{-sr} = 0$, gives us a $(2\pi/\omega, e^{2\pi s0/\omega})$-periodic solution for (2.2).

Example 2.4.2 Let $\phi : \mathbb{R} \to X$ be a X-valued periodic function with period ω. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function with the semigroup property, that is, $\phi(t + s) = \phi(s)\phi(t)$ for all $t, s \in \mathbb{R}$ and such that $\phi(\omega) \neq 0$. Then

$$v(t) = \phi(t)u(t) \tag{1.6}$$

is a (ω, c^{ω})-periodic function. Taking $\phi(t) = e^{i\omega t}$ we obtain a periodic function.

Remark 1.4.9. *In general, if f is a function with the semigroup property such that $f(\omega) = 0$, and if u is a (ω, c)-periodic function, then*

$$v(t) = f(t)u(t)$$

is a $(\omega, cf(\omega))$-periodic function.

Moreover, let $(u_k)_{k\in\mathbb{N}}$ be a sequence of (ω, c)-periodic functions and $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions with the semigroup property such that $f_k(\omega) = 0$ for all $k \in \mathbb{N}$. Then the function

$$v(t) = \sum_{k=1}^{\infty} f_k(t)u_k(t)$$

Functions with the semigroup property and such that $\phi_k(\omega) = p \neq 0$ for all $k \in \mathbb{N}$. Assume that

$$\sum_{k=1}^{\infty} \phi_k(t)\mu_k(t)$$

is a uniformly convergent series on $\mathbb{R}$. Then

$$f(t) = \sum_{k=1}^{\infty} \phi_k(t)\mu_k(t)$$

is a (ω, cp)-periodic function. As a particular case, if the series

$$\sum_{k=1}^{\infty} \phi_k(t)\frac{\cos((2k + 1)t)}{k^2}$$

is uniformly convergent, then

$$f(t) = \sum_{k=1}^{\infty} \phi_k(t) \frac{\cos((2k+1)t)}{k^2}$$

is a $(\pi, -p)$*-periodic function. In this case, calling* $\sigma(p)$ *the sign of* p*, we have*

$$c^*(t) = (-p)^{1/\nu} = e^{\ln|p| + i\sigma(p)\pi} = |p| e^{i\sigma(p)\pi},$$

and hence the π*-periodic part of* f *is*

$$u(t) = \sum_{k=1}^{\infty} |p|^{-1}(1 - i\sigma(p))\phi_k(t) \frac{\cos((2k+1)t)}{k^2},$$

and the $(-p)$*-factorization of* f *is given by*

$$f(t) = c^{\nu}(t)u(t) = |p| \sum_{k=1}^{\infty} \phi_k(t) \frac{\cos((2k+1)t)}{k^2}.$$

Next, we show a convolution theorem.

Theorem Let $f \in P_{\text{vec}}(\mathbb{R}, X)$ with $f(t) = c^{\nu}(t)p(t)$, $p \in P_{\omega}(\mathbb{R}, X)$. If $k^*(t) := c^{\nu}(-t)k(t) \in L^1(\mathbb{R})$, then $k * f \in P_{\text{vec}}(\mathbb{R}, X)$, where

$$(k * f)(t) = \int_{-\infty}^{\infty} k(t-s) f(s)\, ds.$$

Proof. The conclusion follows from the fact that $(k * f)(t) = c^{\nu}(t)(k^* * p)(t)$.

Example 2.1.1 Consider the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = f(x). \end{cases}$$

Let $u(t)$ be a regular solution with $u(x,0) = f(x)$. Then it is known that

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4t}} f(s)\, ds.$$

Fix $t > 0$ and assume that $f(x)$ is (ω, c)-periodic. Then, by Theorem 2.8, $u(x+\omega, t) = cu(x,t)$, hence $u(x,t)$ is (ω, c)-periodic with respect to x.

In order to define a norm over the set $P_{\text{vec}}(\mathbb{R}, X)$ to give it a Banach structure, we need to deal with the following characteristics of their elements: the non-boundedness and its periodicity. The periodicity suggests using a sup-norm, as is possible in $P_{\omega}(\mathbb{R}, X)$, but if $c \neq 1$ and we take $\|f\| = \sup_{t \in \mathbb{R}} \|f(t)\|$, then $\|f\| = \infty$, for all $f \in P_{\text{vec}}(\mathbb{R}, X)$.

The most natural way to avoid the unboundedness of the elements is to restrict the attention to some local bounded case, for example

$$P_{\text{vec}}^{+} := \{ f : \mathbb{R} \to X : f(t+\omega) = cf(t),\ |c| \le 1 \}$$

with the norm

$$\|f\|_\omega := \sup_{t\in[0,\omega]} \|f(t)\|,$$

but it imposes a strong restriction to the study of the (ω, c)-periodic functions. Moreover, the use of norm (2.3) in the space P^+_{vec} (bounded case) implies a loss of periodic structure in the following sense: if we take $f_1(t) := e^{-t}\cos(t)$ and $f_2(t) := e^{-t}\sin(t)$ in P^+_{vec}, with periodic components $\cos(t)$ and $\sin(t)$ respectively, which have the same 2π-period, and belong to $P_{2\pi}(\mathbb{R}, \mathbb{R})$, then f_1 and f_2 must have the same norm. Nevertheless,

$$\|f_1\|_\infty = 1, \quad \|f_2\|_\infty = \frac{e^{-\pi/4}}{\sqrt{2}} < 1.$$

Theorem $P_{\text{vec}}(\mathbb{R}, X)$ is a Banach space with the norm

$$\|f\|_{l\text{vec}} := \sup_{t\in[0,\omega]} \|\theta^\nu(-t)f(t)\|.$$

Proof. Let $\{w_n\}_{n\in\mathbb{N}} \subset P_{\text{vec}}(\mathbb{R}, X)$ be a Cauchy sequence. By Proposition 2.2 we can write $w_n(t) = c^\nu(t)p_n(t)$, where $p_n \in P_\omega(\mathbb{R}, X)$. Also, $\|p_n - p_m\|_\omega = \|w_n - w_m\|_{l\text{vec}}$ implies that $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $P_\omega(\mathbb{R}, X)$, which is a Banach space with respect to the norm $\|\cdot\|_\omega$. Then there exists a ω-periodic function $p(t)$ such that $p_n \to p$ uniformly in $[0, \omega]$, and in consequence, $w_n(t) \to w(t) := c^\nu(t)p(t)$ with the $l$vec-norm in $P_{\text{vec}}(\mathbb{R}, X)$.

If $F \in C(\mathbb{R} \times X, X)$ and $\varphi \in P_{\text{vec}}(\mathbb{R}, X)$, we study the invariance on $P_{\text{vec}}(\mathbb{R}, X)$ for the Nemystkiĭ's operator $\mathcal{N}(\varphi)(t) = F(t, \varphi(t))$.

Theorem Let $F \in C(\mathbb{R} \times X, X)$ and $(\omega, c) \in \mathbb{R}^+ \times (\mathbb{C} \setminus \{0\})$ given. Then the following are equivalent:

1. For every $\varphi \in P_{\text{vec}}(\mathbb{R}, X)$ we have that $\mathcal{N}(\varphi) \in P_{\text{vec}}(\mathbb{R}, X)$;
2. $F(t + \omega, x) = cF(t, x)$ for all $(t, x) \in \mathbb{R} \times X$.

Proof. It is clear that (1) follows immediately from (2). To prove the reciprocal, it is sufficient to consider $\varphi(s) = e^{i(t-s)\cos\left(\frac{2\pi(t-s)}{\omega}\right)}$, which is in $P_{\text{vec}}(\mathbb{R}, X)$ and $\varphi(t) = cx$.

Example 2.3.1 The following functions F satisfy the hypothesis (2) in Theorem 2.11.

1. The function $F(t, u) = f(t)g(u)$ for all $t \in \mathbb{R}$ and for all $u \in X$ where f is a (ω, c)-periodic function and g is a multiplicative function (i.e., $g(ab) = g(a)g(b)$ for all $a, b \in R$ with $g(c) \neq 0$.
2. The function $F(t, x) = f(t)g(x)$ for all $t \in \mathbb{R}$ and for all x

1.5 (ω, c)-Pseudo Periodic Functions

Définition 2.5. A function $f \in C(\mathbb{R}, X)$ is said to be (ω, c)-pseudo periodic if $f = g + h$ where $g \in P_{\omega,c}(\mathbb{R}, X)$ and $h \in AA_{0,c}(X)$. The collection of those functions (with the same c-period ω for the first component) will be denoted by $PP_{\omega,c}(X)$.

Remark 2.6. The preceding collection includes the pseudo periodic functions

$$PP_{\omega,1}(X) := \{f \in C(\mathbb{R}, X) : f = g + h,\ g \in P_{\omega,1}(\mathbb{R}, X),\ h \in AA_0(X)\},$$

the pseudo anti-periodic functions

$$PP_{\omega,(-1)}(X) := \{f \in C(\mathbb{R}, X) : f = g + h,\ g \in P_{\omega,(-1)}(\mathbb{R}, X),\ h \in AA_0(X)\},$$

and pseudo (ω, c)-Bloch-periodic functions

$$PP_{\omega,e^{ik\omega}}(X) := \{f \in C(\mathbb{R}, X) : f = g + h,\ g \in P_{\omega,e^{ik\omega}}(\mathbb{R}, X),\ h \in AA_0(X)\}.$$

Example 2.7. Let

$$\varphi(t) = \max_{k\in\mathbb{Z}} e^{-(t\pm k^2)^2}, \quad t \in \mathbb{R}.$$

It follows from [21] that $\varphi \in AA_0(\mathbb{R}, \mathbb{R})$. Let

$$f_1(t) = \sin t + \varphi(t), \quad t \in \mathbb{R}.$$

Then f_1 is pseudo periodic because $g(t) = \sin t$ is periodic with period 2π and pseudo anti-periodic because $g(t) = \sin t$ is anti-periodic (with antiperiod π). Analogously, the function

$$f_2(t) = e^{ikt} + \varphi(t), \quad t \in \mathbb{R}$$

belongs to $PP_{\omega,e^{ik\omega}}(\mathbb{R}, \mathbb{R})$. The same is true for any $\varphi \in AA_{0,c}(\mathbb{R})$.

The following proposition gives a characterization of the (ω, c)-pseudo periodic functions.

Proposition 2.8 Let $f \in C(\mathbb{R}, X)$. Then f is (ω, c)-pseudo periodic if and only if

$$f(t) = c^{\wedge}(t)u(t), \quad c^{\wedge}(t) = c^{t/\omega}, u \in PP_{\omega}(X). \tag{1.7}$$

Proof of the proposition 2.8

It is clear that if f satisfies (1.1), then f is a (ω, c)-pseudo periodic function. In order to show the inverse statement, let $f \in PP_{\omega,c}(X)$. Then there exist $g \in P_{\omega,c}(\mathbb{R}, X)$ and $h \in AA_{0,c}(X)$ such that $f = g + h$. If we write

$$u(t) := c^{\wedge(-t)} f(t) = c^{-t/\omega} f(t),$$

then

$$\nu(t) = c^{\wedge(-t)} g(t) + c^{\wedge(-t)} h(t) =: F_1(t) + F_2(t).$$

It follows from [45, Proposition 2.5] that $F_1 \in P_{\omega}(\mathbb{R}, X)$, and by definition of $AA_{0,c}(X)$, we have that $F_2 \in AA_0(X)$. Hence, $\nu \in PP_{\omega}(X)$.

Remark The decomposition in Definition 2.5 is unique, that is, there exist a unique $g \in P_{\omega,c}(\mathbb{R}, X)$ and a unique $h \in AA_{0,c}(X)$ such that $f = g + h$. Indeed, suppose that

$$f(t) = g_1(t) + h_1(t) = g_2(t) + h_2(t), \quad g_1, g_2 \in P_{\omega,c}(\mathbb{R}, X), \quad h_1, h_2 \in AA_{0,c}(X), \quad t \in \mathbb{R}.$$

Then

$$u(t) := c^{\wedge(-t)} f(t) = c^{\wedge(-t)} g_1(t) + c^{\wedge(-t)} h_1(t) = c^{\wedge(-t)} g_2(t) + c^{\wedge(-t)} h_2(t).$$

By Proposition 2.2, ν belongs to $PP_\omega(X)$. By the unique representation of the functions in this space, we have that

$$c^{\wedge(-t)} g_1(t) = c^{\wedge(-t)} g_2(t), \quad c^{\wedge(-t)} h_1(t) = c^{\wedge(-t)} h_2(t),$$

and consequently,

$$g_1(t) = g_2(t), \quad h_1(t) = h_2(t), \quad \forall t \in \mathbb{R}.$$

Remark Note that if $|c| \geq 1$, then $AA_0(X) \subset AA_{0,c}(X)$, and consequently,

$$P_{\omega,c}(\mathbb{R}, X) + AA_0(X) \subset PP_{\omega,c}(X).$$

Lemma Let $\alpha \in \mathbb{C}$. Then:

(a) $(f + g) \in PP_{\omega,c}(X)$ and $\alpha h \in PP_{\omega,c}(X)$ whenever $f, g, h \in PP_{\omega,c}(X)$.

(b) If $\tau \in \mathbb{R}$, then $f_\tau(t) = f(t + \tau) \in PP_{\omega,c}(X)$ whenever $f \in PP_{\omega,c}(X)$.

Proof: The proof of (a) is a consequence of the definition. (b) follows from the invariant property of the space $P_{\omega,c}(\mathbb{R}, X)$ and Lemma 2.16.

Example 1 Let

$$\varphi(t) := t|\sin t|^N, \quad \text{for } N > 6.$$

From [4, Example, p. 1143], we have that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(s)|\, ds = 0,$$

and $\varphi(t) \to \infty$ at the points $t = \frac{1}{2} + k$ as $|k| \to \infty$. Let

$$f(t) = 2t \sin t + bt\varphi(t), \quad t \in \mathbb{R}, \quad |b| \leq 2.$$

Then $f \in PP_{\pi,-2\pi}(\mathbb{R})$. Indeed, note that $g(t) := 2t \sin t$ is $(\pi, -2\pi)$-periodic. Let us prove that $h(t) := bt\varphi(t)$ belongs to $AA_{0,-2\pi}(\mathbb{R})$.

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|(-2\pi)^{\wedge(-s)} h(s)\right| ds = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |bs\varphi(s)|\, ds \leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(s)| ds = 0.$$

Hence, f is a (ω, c)-pseudo periodic function which is not a (ω, c)-asymptotically periodic function.

Proof

It is clear that if f satisfies (1.1) then f is a (ω, c)-pseudo periodic function. In order to show the inverse statement, let $f \in PP^c_\omega(X)$. Then there exist $g \in P^c_\omega(\mathbb{R}, X)$ and $h \in AA^c_0(X)$ such that $f = g + h$. If we write $u(t) := c^\wedge(-t)f(t) = c^{-t/\omega} f(t)$, then $u(t) = c^\wedge(-t)g(t) + c^\wedge(-t)h(t) =: F_1(t) + F_2(t)$. It follows from [45],(proposition2.8) that $F_1 \in P_\omega(\mathbb{R}, X)$ and by definition of $AA^c_0(X)$ we have that $F_2 \in AA_0(X)$. Hence $u \in PP_\omega(X)$. where $M\|h\|$ denotes the mean of the norm of h.

Example 2 Let $X = \mathbb{C}$,$|b| \leq 2$. Consider
$f(t) = 2t \sin t + bth(t), \quad t \in \mathbb{R}$,
where h satisfies one of the following conditions:

(a) is integrable, or

(b) L^p-integrable for $1 < p < \infty$, or

(c) asymptotic at t in $-\infty$ and ∞.

Then f is a $(\pi, -2\pi)$-pseudo periodic function. Since

$$c^\wedge(t) = \exp\left(\frac{t}{\pi}\log(-2\pi)\right) = 2^t e^{it},$$

then by Proposition 2.2 we have that

$$g(t) = 2te^{it}u_1(t),$$

where

$$u_1(t) = \sin t(\cos t - i\sin t)$$

is periodic with period $\omega = \pi$. Analogously,

$$bth(t) = 2te^{it}u_2(t),$$

where

$$u_2(t) = \frac{b}{2}th(t)(\cos t - i\sin t)$$

belongs to $AA_0(X)$. Hence, f has the decomposition

$$f(t) = 2t\sin t + bth(t) = 2t(\cos t + i\sin t)\left[\sin t(\cos t - i\sin t) + \frac{b}{2}th(t)(\cos t - i\sin t)\right].$$

Example 3 Let $u : \mathbb{R} \to X$ be an X-valued periodic function with period ω and $v : \mathbb{R} \to X$ in $AA_0(X)$. Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a function with the semigroup property, that is,

$$\varphi(t + s) = \varphi(s)\varphi(t) \quad \text{for all } t, s \in \mathbb{R},$$

and such that $\varphi(\omega) \neq 0$. Then

$$z(t) = \varphi(t)u(t) + \varphi(t)v(t), \quad t \in \mathbb{R},$$

is a $(\omega, \varphi(\omega))$-asymptotically periodic function if $\phi(t) := [\varphi(\omega)]^\wedge(-t)\varphi(t)$
is bounded. As a particular case, we take $\varphi(t) = e^{ikt}$ and obtain the pseudo periodic Bloch functions.

Remark In general, if u is a (ω, c)-pseudo periodic function and φ is a function with the semigroup property such that $\varphi(\omega) \neq 0$, then

$$z(t) := \varphi(t)u(t)$$

is a $(\omega, c\varphi(\omega))$-pseudo periodic function if

$$\phi(t) := [\varphi(\omega)]^\wedge(-t)\varphi(t)$$

is bounded. Moreover, let $(u_k)_{k\in\mathbb{N}}$ be a sequence of (ω, c)-pseudo periodic functions and $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence of functions with the semigroup property.Assume that

$$\sum_{k=1}^{\infty} \varphi_k(t)u_k(t)$$

is a uniformly convergent series on $\mathbb{R}$. Then

$$f(t) = \sum_{k=1}^{\infty} \varphi_k(t)u_k(t)$$

is a (ω, c_p)-pseudo periodic function if $\varphi_k(t) := p \wedge (-t)\varphi_k(t)$ is bounded for $k \in \mathbb{N}$, and such that $\varphi_k(\omega) = p \neq 0$ for all $k \in \mathbb{N}$.

Lemma 2.16

$AA_{0,c}(X)$ is translation invariant, and for every $h \in AA_0(X)$, we have that

$$M_c(g+h) = M_c(g) \quad \text{for all } g \in C(\mathbb{R}, X).$$

Proof

Let $h \in AA_{0,c}(X)$ and $\tau \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T} c \wedge (-\sigma)h(\sigma-\tau)\,d\sigma &= \frac{1}{2T}\int_{-T-\tau}^{T+\tau} c \wedge (-u-\tau)h(u)\,du \\ &\leq c \wedge (-\tau)\frac{1}{2T}\int_{-T-|\tau|}^{T+|\tau|} c \wedge (-u)h(u)\,du \\ &= c \wedge (-\tau)(T+|\tau|)\frac{1}{2(T+|\tau|)}\int_{-T-|\tau|}^{T+|\tau|} c \wedge (-u)h(u)\,du.\end{aligned}$$

Taking the limit as $T \to \infty$, the expression tends to 0. The last assertion follows from the linearity of M.

We recall (see [9]) that the norm in the space $P_{\omega c}(\mathbb{R}, X)$ is given by

$$\|f\|_{\omega c} := \sup_{t\in[0,\omega]} |c| \wedge (-t)f(t).$$

Proposition 2.17

Let $f \in P_{\omega c}(\mathbb{R}, X)$. Then the range $\{c \wedge (-t)f(t) : t \in \mathbb{R}\}$ is relatively compact in X, that is, given $\varepsilon > 0$, for all $t \in \mathbb{R}$, there exist $x_1, \ldots, x_k \in X$ such that

$$\|c \wedge (-t)f(t) - x_i\| < \varepsilon \quad \text{for some } i = 1, \ldots, k.$$

Theorem 2.18

$PP_{\omega c}(X)$ is a Banach space with the norm

$$\|f\|_{p\omega c} := \sup_{t\in\mathbb{R}} |c| \wedge (-t)f(t).$$

Proof

Let (f_n) be a Cauchy sequence in $PP_{\omega c}(X)$. Then, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$, we have

$$\|f_n - f_m\|_{p\omega c} < \varepsilon.$$

Since $f_m, f_n \in PP_{\omega c}(X)$, Proposition 2.8 implies that there exist $u_m, u_n \in PP_\omega(X)$ such that

$$f_m(t) = c \wedge (t)u_m(t) \quad \text{and} \quad f_n(t) = c \wedge (t)u_n(t).$$

Now, note that for $m, n \geq N$

$$\begin{aligned}\|u_m - u_n\|_{p\omega} &= \sup_{t\in\mathbb{R}} |u_m(t) - u_n(t)| \\ &= \sup_{t\in\mathbb{R}} |c| \wedge (-t)(f_m(t) - f_n(t)) \\ &= \|f_n - f_m\|_{p\omega c} < \varepsilon.\end{aligned}$$

It follows that (u_n) is a Cauchy sequence in $PP_\omega(X)$. Since $PP_\omega(X)$ is complete, then there exists $u \in PP_\omega(X)$ such that $\|u_n - u\|_{p\omega} \to 0$ as $n \to \infty$. Let us define $f(t) := c \wedge (t)u(t)$. We claim that $\|f_n - f\|_{p\omega c} \to 0$ as $n \to \infty$. Indeed,

$$\begin{aligned}\|f_n - f\|_{p\omega c} &= \sup_{t\in\mathbb{R}} |c| \wedge (-t)(f_n(t) - f(t)) \\ &= \sup_{t\in\mathbb{R}} |c| \wedge (-t)c \wedge (t)(u_n(t) - u(t)) \\ &= \sup_{t\in\mathbb{R}} |u_n(t) - u(t)| \to 0 \quad (n \to \infty).\end{aligned}$$

Hence, $PP_{\omega c}(X)$ is a Banach space with the norm $\|\cdot\|_{p\omega c}$.

Theorem 2.19 (see [[45], Theorem 2.7])
Let $f \in P_{\omega c}(\mathbb{R}, X)$ with $f(t) = c \wedge (t)p(t)$, $p \in P_\omega(\mathbb{R}, X)$. If

$$k^\sim(t) := c \wedge (-t)k(t) \in L^1(\mathbb{R}),$$

then

$$(k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\,ds \in P_{\omega c}(\mathbb{R}, X).$$

Lemma 2.20 Assume that $k \sim (t) := c \wedge (-t)k(t) \in L^1(\mathbb{R})$. Then $h \in AA_{0,c}(X)$ implies that $k * h \in AA_{0,c}(X)$.

Proof It is clear that the convolution $k * h$ is a continuous function. Then

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T} c \wedge (-t)(k*h)(t)\,dt &\leq \frac{1}{2T}\int_{-T}^{T} |c \wedge (-t)| \int_{-\infty}^{\infty} |k(t-s)||h(s)|\,ds\,dt \\ &= \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} |k \sim (t-s)||c \wedge (-s)h(s)|\,ds\,dt \\ &= \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} |k \sim (s)||c \wedge (-(t-s))h(t-s)|\,ds\,dt \\ &= \int_{-\infty}^{\infty} |k \sim (s)|\Phi_T(s)\,ds,\end{aligned}$$

where

$$\Phi_T(s) := \frac{1}{2T}\int_{-T}^{T} |c \wedge (-(t-s))h(t-s)|\,dt.$$

Since $AA_{0,c}(X)$ is translation invariant by Lemma 2.16, then $\Phi_T(s) \to 0$ as $T \to \infty$. Next, since Φ_T is bounded ($|\Phi_T| \leq |h|_{p\omega c}$) and $k \sim\in L^1(\mathbb{R})$, using the dominated convergence theorem, it follows that

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} |k \sim (s)| \Phi_T(s)\, ds = 0.$$

Hence $k * h \in AA_{0,c}(X)$.

Theorem 2.21 Let $f \in PP_{\omega c}(X)$ with $f(t) = c \wedge (t)p(t)$, $p \in PP_\omega(X)$. If for some $k(t)$ we have that $k \sim (t) := c \wedge (-t)k(t) \in L^1(\mathbb{R})$, then

$$(k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\, ds = c \wedge (t)(k \sim *p)(t).$$

In particular, $(k * f)(t) \in PP_{\omega c}(X)$.

Example 2.22 Consider the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & t > 0, x \in \mathbb{R}, \\ u(x,0) = f(x). \end{cases}$$

Let $u(x,t)$ be a regular solution with $u(x,0) = f(x)$. Then it is known that

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4t}} f(s)\, ds, \quad t > 0, x \in \mathbb{R}.$$

Fix $t_0 > 0$ and assume that $f(x)$ is (ω, c)-pseudo periodic. Then, by Theorem 2.21, we have that $u(x, t_0)$ is (ω, c)-pseudo periodic with respect to x.

Lemma 2.23 Let $h \in C(\mathbb{R}, X)$ such that $\sup_{t\in\mathbb{R}} |c \wedge (-t)h(t)| < \infty$. Then $h \in AA_{0,c}(X)$ if and only if

$$(\forall \epsilon > 0) \lim_{T\to\infty} \frac{1}{2T} \operatorname{meas}(M_{T,\epsilon}(h)) = 0.$$

where

$$M_T, \mathfrak{h} := \left\{ t \in [-T,T] : \int_{-t}^{t} c \wedge (-t)h(t)\, dt \geq \mathfrak{h} \right\}$$

.

Proof: Assume that $h \in \mathcal{A}_{A0,c}(X)$ and suppose that there exists $\mathfrak{h}_0 > 0$ such that

$$\frac{1}{2T} \cdot \operatorname{meas}(M_{T,\mathfrak{h}_0}(h)) \not\to 0 \quad \text{as} \quad T \to \infty.$$

That is, there exists $\delta > 0$ such that, for $n \in \mathbb{N}$,

$$\frac{1}{2T_n} \cdot \operatorname{meas}(M_{T_n,\mathfrak{h}_0}(h)) \geq \delta \quad \text{for} \quad T_n > n.$$

Then,

$$\frac{1}{2T_n} \int_{-T_n}^{T_n} c \wedge (-t)h(t)\, dt = \frac{1}{2T_n} \int_{M_{T_n,\mathfrak{h}_0}} c \wedge (-t)h(t)\, dt + \frac{1}{2T_n} \int_{[-T_n,T_n]\setminus M_{T_n,\mathfrak{h}_0}} c \wedge (-t)h(t)\, dt.$$

We have the inequality

$$\frac{1}{2T_n}\int_{M_{T_n,\mathfrak{h}_0}} c\wedge(-t)h(t)\,dt \geq \frac{1}{2T_n}\cdot \text{meas}(M_{T_n,\mathfrak{h}_0}(h))\cdot \mathfrak{h}_0 \geq \delta\mathfrak{h}_0,$$

which leads to a contradiction.

Now, assume (2.3). We prove that $h \in \mathcal{A}_{A0,c}(X)$. By (2.3), we have that there exists $M > 0$ such that $c\wedge(-t)h(t) \leq M$, and for all $\mathfrak{h} > 0$, there exists $T_0 > 0$ such that for $T > T_0$, we have

$$\frac{1}{2T}\cdot \text{meas}(M_{T,\mathfrak{h}}(h)) < M+1.$$

Then,

$$\frac{1}{2T}\int_{-T}^{T} c\wedge(-t)h(t)\,dt = \frac{1}{2T}\int_{M_{T,\mathfrak{h}}} c\wedge(-t)h(t)\,dt + \frac{1}{2T}\int_{[-T,T]\setminus M_{T,\mathfrak{h}}} c\wedge(-t)h(t)\,dt.$$

We can estimate:

$$\frac{1}{2T}\int_{M_{T,\mathfrak{h}}} c\wedge(-t)h(t)\,dt \leq \frac{1}{2T}M\cdot \text{meas}(M_{T,\mathfrak{h}}(h)),$$

and

$$\frac{1}{2T}\left(2T - \text{meas}(M_{T,\mathfrak{h}}(h))\right) < (M-1)\mathfrak{h} + M + 1 + \mathfrak{h} < 2\mathfrak{h}.$$

Hence, $h \in \mathcal{A}_{A0,c}(X)$.

Next, we have the following composition result. The idea of the proof follows from [21][Theorem 2.4].

Theorem 2.24: Let $f(t,x) = g(t,x) + h(t,x)$ where $g(t+\omega, cx) = cg(t,x)$ and $h \in \mathcal{A}_{A0,c}(X,X)$. Assume

(a) $\sup_{t\in\mathbb{R}} |c\wedge(-t)f(t,x)| < \infty$ for all $x \in X$,

(b) $f_t(z) := c\wedge(-t)f(t, c\wedge(t)z)$ is uniformly continuous for z in any bounded subset $K \subset X$, uniformly in $t \in \mathbb{R}$; that is, given $\mathfrak{h} > 0$ and $K \subset X$ bounded, there exists $\delta > 0$ such that $x, y \in K$ and $|x - y| < \delta$ imply that

$$|f_t(x) - f_t(y)| \leq \mathfrak{h} \quad \text{for all} \quad t \in \mathbb{R}.$$

Let $h_t(x) := c\hat{}(-t)h(t, c\hat{}(t)x)$. Then $h_t(x)$ is uniformly continuous for x in any bounded set of X, uniformly in $t \in \mathbb{R}$, and

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} h_t(x)\,dt = 0$$

for x in any bounded subset of X.

If $\varphi \in PP\omega_c(X)$, then $f(\cdot, \varphi(\cdot)) \in PP\omega_c(X)$.

Proof: Let $\varphi(t) = \alpha(t) + \beta(t)$, where $\alpha \in P\omega_c(\mathbb{R}, X)$ and $\beta \in AA0, c(X)$. Then we have

$$f(t, \varphi(t)) = [f(t, \varphi(t)) - f(t, \alpha(t))] + g(t, \alpha(t)) + h(t, \alpha(t)) := F(t) + G(t) + H(t).$$

By [[45], Theorem 2.11], we have that $G(t) = g(t, \alpha(t))$ belongs to $P\omega_c(\mathbb{R}, X)$.

On the other hand, note that $\phi(t) := c^\wedge(-t)\varphi(t)$ and $\phi_1(t) := c^\wedge(-t)\alpha(t)$ are bounded by definition and Proposition 2.17, respectively. From here, we can choose $K \subset X$ bounded such that $\phi([-T, T])$, $\phi_1([-T, T]) \subset K$. Under assumption (b), $c^\wedge(-t)f(t, c^\wedge(t)\cdot)$ is uniformly continuous on the bounded set K, uniformly for $t \in [-T, T]$, so given $\epsilon > 0$, there exists $\delta := \delta_{\epsilon,K}$ such that

$$\|\phi(t) - \phi_1(t)\| = \|c^\wedge(-t)\varphi(t) - c^\wedge(-t)\alpha(t)\| = \|c^\wedge(-t)\beta(t)\| \leq \delta$$

implies that

$$\|c^\wedge(-t)f(t, \varphi(t)) - c^\wedge(-t)f(t, \alpha(t))\| = \|c^\wedge(-t)F(t)\| \leq \epsilon$$

for all $t \in [-T, T]$.

Then we have

$$\lim_{T\to\infty} \frac{1}{2T}\text{meas}(M_{T,\epsilon}(F)) \leq \lim_{T\to\infty} \frac{1}{2T}\text{meas}(M_{T,\delta}(\beta)).$$

Since $\beta \in AA0, c(X)$, Lemma 2.3 yields for the above δ that

$$\lim_{T\to\infty} \frac{1}{2T}\text{meas}(M_{T,\delta}(\beta)) = 0.$$

From here, we conclude that $F \in AA0, c(X)$.

Finally, we prove that $H \in AA0, c(X)$. Let $\phi(t) := c^\wedge(-t)\alpha(t)$ and $I = \phi([-T, T])$. Then ϕ is uniformly continuous on $[-T, T]$, and therefore I is compact in X. Let $\epsilon > 0$. Then, for every $\delta = \delta(\epsilon) > 0$, there exist finite open balls O_k (for $k = 1, 2, \ldots, m$) with centers $x_k \in I$, respectively, such that $I \subset \bigcup_{k=1}^m O_k$.

Then, by the uniform continuity of $c^\wedge(-t)h(t, c^\wedge(t)\cdot)$, we have that

$$\|c^\wedge(-t)h(t, \alpha(t)) - c^\wedge(-t)h(t, c^\wedge(t)x_k)\| < \frac{\epsilon}{2}, \quad t \in [-T, T].$$

The set $B_k := \{t \in [-T, T] : \phi(t) \in O_k\}$ is open in $[-T, T]$, and $[-T, T] = \bigcup_{k=1}^m B_k$. Let

$$E_1 = B_1, \quad E_k := B_k \setminus \bigcup_{j=1}^{k-1} B_j \quad (k = 2, \ldots, m).$$

Then $E_i \cap E_j = \emptyset$ when $i \neq j$, $1 \leq i, j \leq m$. Note that

$$\left\{t \in [-T, T] :\, c^\wedge(-t)h(t, \alpha(t)) \geq \frac{\beta}{2}\right\} \subset \bigcup_{k=1}^m \left\{t \in [-T, T] :\, |h(t, \alpha(t)) - h(t, c^\wedge(t)x_k)| \geq \frac{\beta}{2|c^\wedge(-t)|}\right\}.$$

It follows from (2.4) that $\{t \in [-T, T] : c^\wedge(-t)\left[h(t, \alpha(t)) - h(t, c^\wedge(t)x_k)\right] \geq \frac{\beta}{2}\}$ are empty for $k = 1, \ldots, m$. Therefore,

$$\frac{1}{2T}\text{meas}\left(M_{T,\beta/2}h(t, \alpha(t))\right) \leq \prod_{k=1}^m \frac{1}{2T}\text{meas}\left(M_{T,\beta/2}h(t, c^\wedge(t)x_k)\right).$$

Since $h(t, c^\wedge(t)x_k) \in AA_0, c(X, X)$ by (c), we have that

$$\lim_{T\to\infty} \frac{1}{2T}\text{meas}\left(M_{T,\beta/2}h(t, c^\wedge(t)x_k)\right) = 0 \text{ for all } k = 1, \dots, m;$$

and therefore,

$$\lim_{T\to\infty} \frac{1}{2T}\text{meas}\left(M_{T,\beta/2}h(t, \alpha(t))\right) = 0,$$

that is, $H \in AA_0, c(X)$. To summarize, we have proved that $f(\cdot, \varphi(\cdot)) \in PP_{\omega c}(X)$.

Next, we present another composition theorem.

Theorem 2.25: Let $f(t, x) = g(t, x) + h(t, x)$, where $g(t + \omega, cx) = cg(t, x)$ and $h \in AA_0, c(X, X)$. Assume the following:

(a) $h_t(x) := c^\wedge(-t)h(t, c^\wedge(t)x)$ is uniformly continuous for x in any bounded set of X uniformly in $t \in \mathbb{R}$ and

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} c^\wedge(-t)h(t, c^\wedge(t)x)\, dt = 0 \text{ for } x \in \text{any bounded subset of } X.$$

(b) There exists a nonnegative bounded function $L_f(t)$ such that

$$|f(t, x) - f(t, y)| \leq L_f(t)\|x - y\|, \quad t \in \mathbb{R},\, x, y \in X. \tag{2.5}$$

If $\varphi \in PP_{\omega c}(X)$, then $f(\cdot, \varphi(\cdot)) \in PP_{\omega c}(X)$. Let $\varphi(t) = \alpha(t) + \beta(t)$ with $\alpha \in P_{\omega c}(\mathbb{R}, X)$ and $\beta \in AA_{0,c}(X)$. Then we have

$$f(t, \varphi(t)) = f(t, \varphi(t)) - f(t, \alpha(t)) + g(t, \alpha(t)) + h(t, \alpha(t)) := F(t) + G(t) + H(t).$$

Note that

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} |c \wedge (-t)|\, F(t)\, dt &\leq \frac{1}{2T}\int_{-T}^{T} |c \wedge (-t)|\, (f(t, \varphi(t)) - f(t, \alpha(t)))\, dt \\ &\leq \frac{1}{2T}\int_{-T}^{T} |c \wedge (-t)|\, L_f(t)\, |\varphi(t) - \alpha(t)|\, dt. \end{aligned}$$

As $T \to \infty$, this tends to zero, where we have used that $L_f(t) \leq L_f$ and the fact that $\beta \in AA_{0,c}(X)$. It follows that $F \in AA_{0,c}(X)$.

On the other hand, by ([45] , Theorem 2.11), we have that $G(t) = g(t, \alpha(t))$ belongs to $P_{\omega c}(\mathbb{R}, X)$.

Finally, we prove that $H \in AA_{0,c}(X)$. From Proposition 2.17, we have that $K := \{c \wedge (-t)\alpha(t) : t \in \mathbb{R}\}$ is relatively compact in X. Let $\varepsilon > 0$. Then, for every $\delta > 0$, there exist $x_1, \dots, x_k \in I$ such that

$$\{c \wedge (-t)\alpha(t) : t \in \mathbb{R}\} \subseteq \bigcup_{j=1}^{k} B(x_j, \delta).$$

Consequently, given $t \in \mathbb{R}$, we can choose $j \in \{1, \dots, k\}$ such that

$$c \wedge (-t)\alpha(t) - x_j < \delta.$$

Since $h_t(\cdot) = c \wedge (-t)h(t, c \wedge (t)\cdot)$ is uniformly continuous on K uniformly for $t \in \mathbb{R}$, then taking $\delta = \delta(\varepsilon/2)$, we obtain that

$$c \wedge (-t)\left(h(t, c \wedge (t) c \wedge (-t)\alpha(t)) - h(t, c \wedge (t)x_j)\right) < \frac{\varepsilon}{2}, \quad \text{uniformly for } t \in \mathbb{R}.$$

From here, we can conclude that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c \wedge (-t)\left(h(t, \alpha(t)) - h(t, c \wedge (t)x_j)\right)\, dt < \frac{\varepsilon}{2}.$$

On the other hand, since

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c \wedge (-t)h(t, c \wedge (t)\cdot)\, dt = 0$$

on the bounded subsets of X, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c \wedge (-t)h(t, c \wedge (t)x_j)\, dt = 0.$$

Thus, there exists $N \in \mathbb{N}$ such that for all $t \geq N$ we have that

$$\frac{1}{2T} \int_{-T}^{T} c \wedge (-t)h(t, c \wedge (t)x_j)\, dt < \frac{\varepsilon}{2}.$$

Next, for all $t \geq N$ and some $j = 1, 2, \ldots, k$, we have

$$\begin{aligned} \frac{1}{2T} \int_{-T}^{T} c \wedge (-t)\, h(t, \alpha(t))\, dt &\leq \frac{1}{2T} \int_{-T}^{T} c \wedge (-t)\left(h(t, \alpha(t)) - h(t, c \wedge (t)x_j)\right)\, dt \\ &\quad + \frac{1}{2T} \int_{-T}^{T} c \wedge (-t)\, h(t, c \wedge (t)x_j)\, dt \\ &< \varepsilon. \end{aligned}$$

Hence,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c \wedge (-t)H(t)\, dt = 0.$$

Consequently, $f(\cdot, \varphi(\cdot)) \in PP_{\omega c}(X)$.

Chapter 2

Existence and Uniqueness of (ω, c)-Periodic Solutions of Semilinear Evolution Equations

Sommaire

2.1 Preliminary Results

In this paper, we study the homogeneous linear equation:

$$\begin{cases} y'(t) = Ay(t) \ t \in [0, d], \\ y(0) = y_0. \end{cases} \tag{2.1}$$

and the nonhomogeneous linear non-instantaneous impulsive equation:

$$\begin{cases} y'(t) = Ay(t) + g(t) \ t \in [0, d], \\ y(0) = y_0. \end{cases} \tag{2.2}$$

and the nonlinear equation

$$\begin{cases} y'(t) = Ay(t) + g(t, y(t)) \ t \in [0, d], \\ y(0) = y_0. \end{cases} \tag{2.3}$$

2.2 Homogeneous Linear Problem

Let X be a complex Banach space with a norm $\|\cdot\|$.

We denote by $\Upsilon_{\omega,c}$ the set of all continuous and (ω, c)-periodic functions $g : \mathbb{R} \to X$, and

$$\Upsilon^1_{\omega,c} = \Upsilon_{\omega,c} \cap C^1(\mathbb{R}, X).$$

Let us consider first the linear equation

$$\dot{y} = Ay \tag{1}$$

for a continuous linear mapping $A \in \mathcal{L}(X)$. Then its solution is of the form $y(t) = e^{At}y_0$, $t \in \mathbb{R}$, which is (ω, c)-periodic if and only if it holds

$$y(t+\omega) = cy(t) \quad \Leftrightarrow \quad e^{A(t+\omega)}y_0 = ce^{At}y_0 \quad \Leftrightarrow \quad e^{At}e^{A\omega}y_0 = ce^{At}y_0$$

$$\Leftrightarrow \quad e^{A\omega}y_0 = cy_0 \quad \Leftrightarrow \quad (cI - e^{A\omega})y_0 = 0.$$

In this paper, we consider the case when c does not belong to the spectrum $\sigma(e^{A\omega})$ of $e^{A\omega}$, so by the spectral mapping theorem we suppose:

(A1) $c \neq e^{\omega\lambda}$ for all $\lambda \in \sigma(A)$.

2.3 Nonhomogeneous linear Problem

In this section, we study the existence of (ω, c)-periodic solutions of (2.2)

$$\dot{y} = Ay + f(t) \tag{2}$$

for $f \in \Upsilon_{\omega,c}$. We present the following simple observation.

Lemma 2.3.1. *$y \in \Upsilon_{\omega,c}$ if and only if it holds*

$$y(\omega) = cy(0). \tag{3}$$

Proof Clearly, $y \in \Upsilon_{\omega,c}$ implies 3.

On the other hand, if 3 holds for some $y_0 \in C([0,\omega], X)$, then we set

$$y(t) = c^k y_0(t - k\omega), \quad \text{for } t \in [k\omega, (k+1)\omega],\ k \in \mathbb{Z}.$$

Note that

$$\lim_{t\to k\omega^+} y(t) = c^k y_0(0) = c^{k-1}y_0(\omega) = \lim_{t\to k\omega^-} y(t),$$

so the function y is continuous on $\mathbb{R}$, and by construction,

$$y(t+\omega) = c^{k+1}y_0(t+\omega-(k+1)\omega) = c \cdot c^k y_0(t-k\omega) = cy(t),$$

for all $t \in [k\omega, (k+1)\omega]$. Hence, $y \in \Upsilon_{\omega,c}$.

So $y(t)$ is well-defined and continuous. Next, for $t \in [k\omega, (k+1)\omega]$ and $k \in \mathbb{Z}$, we have $t + \omega \in [(k+1)\omega, (k+2)\omega]$, and

$$y(t+\omega) = c^{k+1}y_0(t+\omega-(k+1)\omega) = c^{k+1}y_0(t-k\omega) = c \cdot c^k y_0(t-k\omega) = cy(t).$$

So $y \in \Upsilon_{\omega,c}$. The proof is finished.

Now, set a Banach space $Z = C([0,\omega], X)$ with the maximum norm

$$\|y\|_0 = \max_{t\in[0,\omega]} \|y(t)\|.$$

Now we are ready to prove the following result.

Lemma 2.3.2. *The solution $y \in Z$ of (2) satisfying equation(3) is given by*

$$y(t) = \int_0^\omega K(t,s)f(s)\,ds,$$

where

$$K(t,s) = \begin{cases} ce^{A(t-s)}(cI - e^{A\omega})^{-1}, & \text{for } s \in [0,t], \\ e^{A(\omega+t-s)}(cI - e^{A\omega})^{-1}, & \text{for } s \in (t,\omega], \end{cases}$$

and I is the identity operator.

Proof. The general solution $y \in Z$ of (2) has the form

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}f(s)\,ds.$$

Using condition (3), we compute:

$$y(\omega) = e^{A\omega}y_0 + \int_0^\omega e^{A(\omega-s)}f(s)\,ds = cy(0) = cy_0.$$

Hence,

$$e^{A\omega}y_0 + \int_0^\omega e^{A(\omega-s)}f(s)\,ds = cy_0 \quad \Rightarrow \quad (cI - e^{A\omega})y_0 = \int_0^\omega e^{A(\omega-s)}f(s)\,ds,$$

$$\Rightarrow \quad y_0 = (cI - e^{A\omega})^{-1}\int_0^\omega e^{A(\omega-s)}f(s)\,ds.$$

Substituting into the expression for $y(t)$, we get:

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}f(s)\,ds$$

$$= e^{At}(cI - e^{A\omega})^{-1}\int_0^\omega e^{A(\omega-s)}f(s)\,ds + \int_0^t e^{A(t-s)}f(s)\,ds$$

Now rewrite the first term:

$$= \int_0^\omega e^{At}e^{A(\omega-s)}(cI - e^{A\omega})^{-1}f(s)\,ds + \int_0^t e^{A(t-s)}f(s)\,ds$$

$$= \int_0^t \left(e^{A(t-s)}(cI - e^{A\omega})^{-1}e^{A\omega} + e^{A(t-s)}\right) f(s)\,ds + \int_t^\omega e^{A(\omega+t-s)}(cI - e^{A\omega})^{-1}f(s)\,ds$$

Factor the first integrand:

$$= \int_0^t \left(ce^{A(t-s)}(cI - e^{A\omega})^{-1}\right) f(s)\,ds + \int_t^\omega e^{A(\omega+t-s)}(cI - e^{A\omega})^{-1}f(s)\,ds$$

Thus,

$$y(t) = \int_0^t K(t,s)f(s)\,ds + \int_t^\omega K(t,s)f(s)\,ds = \int_0^\omega K(t,s)f(s)\,ds.$$

Indeed, since

$$(cI - e^{A\omega})(cI - e^{A\omega})^{-1} = I,$$

it follows that

$$c(cI - e^{A\omega})^{-1} = e^{A\omega}(cI - e^{A\omega})^{-1} + I = (cI - e^{A\omega})^{-1}e^{A\omega} + I.$$

The proof is completed.

2.4 Nonlinear equation Problem

In this section, we study the existence of (ω, c)-periodic solutions of (2.3). for $g \in C(\mathbb{R}\times X, X)$ satisfying the following conditions:

(C1) $g(t+\omega, cy) = cg(t,y)$ for all $t \in \mathbb{R}$ and $y \in X$.

(C2) There exists a constant $L > 0$ such that

$$\|g(t,y_1) - g(t,y_2)\| \le L\|y_1 - y_2\| \quad \text{for all } t \in \mathbb{R},\ y_1, y_2 \in X.$$

We are looking for solutions of equation (**??**) in $\Upsilon_{\omega,c}$. First, we note that (C1) implies that if $y \in \Upsilon_{\omega,c}$, then $g(t,y(t)) \in \Upsilon_{\omega,c}$. Then, by Lemmas 2.2 and 2.3, our task is equivalent to solving the fixed point problem

$$y(t) = \int_0^\omega K(t,s)g(s,y(s))\,ds, \quad y \in Z. \tag{5}$$

To solve equation (5), we define an operator $S : Z \to Z$ by

$$(Sy)(t) = \int_0^\omega K(t,s)g(s,y(s))\,ds,$$

for $y \in Z$. Clearly, S is well-defined. Next, for $y_1, y_2 \in Z$, we compute

$$\|(Sy_1)(t) - (Sy_2)(t)\| \le \int_0^\omega \|K(t,s)(g(s,y_1(s)) - g(s,y_2(s)))\|\,ds$$

$$\le \int_0^\omega \|K(t,s)\| \cdot \|g(s,y_1(s)) - g(s,y_2(s))\|\,ds \le L\int_0^\omega \|K(t,s)\| \cdot \|y_1(s) - y_2(s)\|\,ds$$

$$\le L\|y_1 - y_2\|_0 \int_0^\omega \|K(t,s)\|\,ds.$$

Define

$$M = \max_{t\in[0,\omega]} \int_0^\omega \|K(t,s)\|\,ds. \tag{6}$$

Therefore, we arrive at the inequality

$$\|Sy_1 - Sy_2\|_0 \le LM\|y_1 - y_2\|_0.$$

By the Banach fixed point theorem, we obtain the following result.

Theorem Suppose (A1) and consider equation(4) under conditions (C1) and (C2). If

$$LM < 1 \tag{7}$$

for M given by (6), then equation (4) has a unique (ω, c)-periodic solution y satisfying

$$\|y\|_0 \le \frac{M\|g(\cdot,0)\|_0}{1-LM}. \tag{8}$$

Proof. The uniqueness and existence result follows from the Banach fixed point theorem applied to the operator S, in view of Lemmas 2.2 and 2.3.

Furthermore, using the contractive estimate,

$$\|y\|_0 = \|Sy\|_0 \le LM\|y\|_0 + M\|g(\cdot,0)\|_0,$$

we obtain

$$\|y\|_0(1-LM) \le M\|g(\cdot,0)\|_0 \quad \Rightarrow \quad \|y\|_0 \le \frac{M\|g(\cdot,0)\|_0}{1-LM}.$$

Remark 3.2. We can estimate M as follows:

1. First estimate:

$$\begin{aligned}\int_0^\omega \|K(t,s)\|\,ds &\le |c|\left\|(cI-e^{A\omega})^{-1}\right\|\int_0^t e^{\|A\|(t-s)}\,ds \\ &\quad + \left\|e^{A\omega}(cI-e^{A\omega})^{-1}\right\|\int_t^\omega e^{\|A\|(t-s)}\,ds \\ &= |c|\left\|(cI-e^{A\omega})^{-1}\right\|\cdot\frac{e^{\|A\|t}-1}{\|A\|} \\ &\quad + \left\|e^{A\omega}(cI-e^{A\omega})^{-1}\right\|\cdot\frac{1-e^{\|A\|(t-\omega)}}{\|A\|} \\ &\le \frac{e^{\|A\|\omega}-1}{\|A\|}\cdot\max\left\{|c|\left\|(cI-e^{A\omega})^{-1}\right\|,\left\|e^{A\omega}(cI-e^{A\omega})^{-1}\right\|e^{\|A\|\omega}\right\}.\end{aligned}$$

Thus,

$$M \le \frac{e^{\|A\|\omega}-1}{\|A\|}\max\left\{|c|\left\|(cI-e^{A\omega})^{-1}\right\|,\left\|(cI-e^{A\omega})^{-1}e^{A\omega}\right\|e^{\|A\|\omega}\right\}. \tag{9}$$

2. Second estimate:

$$\begin{aligned}\int_0^\omega \|K(t,s)\|\,ds &\le |c|\int_0^t\left\|e^{A(t-s)}(cI-e^{A\omega})^{-1}\right\|\,ds + \int_t^\omega\left\|e^{A(\omega+t-s)}(cI-e^{A\omega})^{-1}\right\|\,ds \\ &= |c|\int_0^t\left\|e^{As}(cI-e^{A\omega})^{-1}\right\|\,ds + \int_t^\omega\left\|e^{As}(cI-e^{A\omega})^{-1}\right\|\,ds \\ &\le \max\{|c|,1\}\int_0^\omega\left\|e^{As}(cI-e^{A\omega})^{-1}\right\|\,ds.\end{aligned}$$

Hence,

$$M \le \max\{|c|, 1\} \int_0^{\omega} \left\| e^{As}(cI - e^{A\omega})^{-1} \right\| ds. \tag{10}$$

Example We consider the case $X = \mathbb{C}^2$, with $c = -1$, $\omega = \pi$, and

$$A = \begin{pmatrix} 2 & -4 \\ 6 & -8 \end{pmatrix}, \quad \text{and} \quad g(t, y) = (g_1(t, y), g_2(t, y)) = a(\sin t \cos(y_1 + y_2), \cos 2t \sin(y_1 - y_2)),$$

where $y = (y_1, y_2)$.

Since all parameters are real, we restrict to $X = \mathbb{R}^2$. Clearly, condition (C1) holds. The spectrum of A is $\sigma(A) = \{-4, -2\}$, so (A1) is satisfied.

Next, by the Mean Value Theorem for vector-valued functions, we get

$$L = \max_{y \in \mathbb{R}^2,\, t \in \mathbb{R}} \|g_y(t, y)\|,$$

where

$$g_y = \begin{pmatrix} -a \sin t \sin(y_1 + y_2) & -a \sin t \sin(y_1 + y_2) \\ a \cos 2t \cos(y_1 - y_2) & -a \cos 2t \cos(y_1 - y_2) \end{pmatrix}.$$

Furthermore, we have:

$$e^{As}(cI - e^{A\omega})^{-1} = \begin{pmatrix} \frac{2e^{4\pi - 4s}}{1 + e^{4\pi}} & -\frac{3}{2} e^{\pi - 2s} \operatorname{sech} \pi \\ e^{\pi - 2s} \operatorname{sech} \pi & -\frac{2e^{4\pi - 4s}}{1 + e^{4\pi}} \\ \frac{3e^{4\pi - 4s}}{1 + e^{4\pi}} & -\frac{3}{2} e^{\pi - 2s} \operatorname{sech} \pi \\ e^{\pi - 2s} \operatorname{sech} \pi & -\frac{3e^{4\pi - 4s}}{1 + e^{4\pi}} \end{pmatrix}.$$

Now we consider three standard norms on $\mathbb{R}^2$ to estimate L and M.

Case 1. $\|y\|_1 = |y_1| + |y_2|$. Then we derive:

$$L = \max_{y \in \mathbb{R}^2,\, t \in \mathbb{R}} \|g_y(t, y)\|_1 = |a| \max_{y,t} \left(|\sin t \sin(y_1 + y_2)| + |\cos 2t \cos(y_1 - y_2)| \right) = 2|a|.$$

By equation (10), we compute:

$$M \le 1.73883.$$

So condition (7) holds if

$$|a| < \frac{1}{2M} = 0.287549. \tag{11}$$

Case 2. $\|y\|_\infty = \max\{|y_1|, |y_2|\}$. Then we derive:

$$L = \max_{y \in \mathbb{R}^2,\, t \in \mathbb{R}} \|g_y(t, y)\|_\infty = 2|a| \max_t \max\{|\sin t|, |\cos 2t|\} = 2|a|.$$

By equation (10), we have $M \le 1.4907$. Thus, condition (7) holds if

$$|a| < 0.335414. \tag{12}$$

Case 3. $\|y\|_2 = \sqrt{y_1^2 + y_2^2}$. We then derive

$$L = \max_{y \in \mathbb{R}^2, t \in \mathbb{R}} \|g_y(t, y)\|_2 = \sqrt{2|a|} \max_{y \in \mathbb{R}^2, t \in \mathbb{R}} \max\{|\cos 2t \cos(y_1 - y_2)|, |\sin t \sin(y_1 + y_2)|\}.$$

Next, we calculate

$$L = \sqrt{2|a|} \max_{t\in\mathbb{R}} \max \{|\sin t|, |\cos 2t|\} = \sqrt{2|a|},$$

and by equation (10), we have $M \leq 1.40635$. Therefore, condition (7) holds if

$$|a| < 0.502795. \tag{13}$$

Consequently, the best estimate from (11), (12), and (13) is (13), corresponding to the norm $\|\cdot\|_2$, when there is a unique π-antiperiodic solution $y(t)$ which is nonconstant.

2.5 An Existence Result

Now we consider instead condition (C2) the following condition:

(C3) There are constants $g_1 \geq 0$ and $g_2 \geq 0$ such that

$$\|g(t,y)\| \leq g_1 + g_2\|y\| \quad \text{for all } t \in \mathbb{R} \text{ and } y \in X.$$

Then, similarly to the previous case, we derive

$$\|(Sy)(t)\| \leq \int_0^\omega \|K(t,s)g(s,y(s))\|\, ds$$

$$\leq (g_1 + g_2\|y\|_0) \int_0^\omega \|K(t,s)\|\, ds \leq M(g_1 + g_2\|y\|_0).$$

Therefore, from the Schauder fixed point theorem, we obtain the following result:

Theorem 4.1: Let $\dim X < \infty$. Suppose (A1) holds and consider equation (4) under conditions (C1) and (C3). If

$$g_2 M < 1 \tag{14}$$

for M given by equation (6), then equation (4) has a (ω, c)-periodic solution y with

$$\|y\|_0 \leq \frac{Mg_1}{1 - Mg_2}.$$

Proof: Set $B(r_0) = \{y \in Z \mid \|y\|_0 \leq r_0\}$ for

$$r_0 = \frac{Mg_1}{1 - Mg_2}.$$

Then, for any $y \in B(r_0)$, the above computation gives

$$\|Sy\|_0 \leq M(g_1 + g_2\|y\|_0) \leq M(g_1 + g_2 r_0) = r_0.$$

Hence, $S : B(r_0) \to B(r_0)$. The Arzelà-Ascoli theorem implies the compactness of S. Therefore, the Schauder fixed point theorem gives the result. The proof is finished.

Example 4.2: Consider the problem from Example 3.3. Then

$$\|g(t,y)\|_1 = |a| \left(|\sin t \cos(y_1 + y_2)| + |\cos 2t \sin(y_1 - y_2)|\right) \leq 2|a|,$$

so we have $g_1 = 2|a|$ and $g_2 = 0$. Consequently, there is a π-antiperiodic solution y for any $0 \neq a \in \mathbb{R}$ with

$$\|y(t)\|_1 \leq 3.47767|a| \quad \text{for any } t \in \mathbb{R}.$$

Note that when equation (13) holds, we have a unique such solution.

Example 4.3: In this example, we consider the case for $c = -1$, $\omega = \pi$, and $g(t, y) = (g_1(t, y), g_2(t, y)) = (a \sin t(|y_1 + y_2| + 1), a \cos t|y_1 - y_2|)$, and A from Example 3.3. We again consider 3 standard norms on $\mathbb{R}^2$ to estimate g_1, g_2, and M.

Case 1. $\|y\|_1 = |y_1| + |y_2|$. Then we derive

$$\begin{aligned}
\|g(t, y)\|_1 &= |a| \left(|\sin t(|y_1 + y_2| + 1)| + |\cos t \cdot |y_1 - y_2||\right) \\
&\leq |a| \left(|\sin t|(|y_1 + y_2| + 1) + |\cos t||y_1 - y_2|\right) \\
&\leq |a| + |a| \left(|\sin t| + |\cos t|\right) \|y\|_1 \\
&\leq |a| + \sqrt{2}|a|\|y\|_1 \\
&= |a| + 1.41421|a|\|y\|_1.
\end{aligned}$$

Hence $g_1 = |a|$, $g_2 \approx 1.41421|a|$, and M is given in Case 1 of Example 3.3. So condition (14) holds if

$$|a| < 0.406656. \tag{15}$$

Case 2. $\|y\|_\infty = \max\{|y_1|, |y_2|\}$. Then we derive

$$\begin{aligned}
\|g(t, y)\|_\infty &= |a| \max\left\{|\sin t(|y_1 + y_2| + 1)|, |\cos t\, |y_1 - y_2||\right\} \\
&\leq |a| \max\left\{|\sin t| \cdot |y_1 + y_2| + |\sin t|, |\cos t| \cdot |y_1 - y_2|\right\} \\
&\leq |a| \max\left\{|\sin t| \cdot (|y_1 + y_2| + 1), |\cos t| \cdot |y_1 - y_2|\right\} \\
&\leq |a| + 2|a| \max\left\{|\sin t|, |\cos t|\right\} \|y\|_\infty \\
&= |a| + 2|a|\|y\|_\infty.
\end{aligned}$$

Hence $g_1 = |a|$, $g_2 = 2|a|$, and M is given in Case 2 of Example 3.3. So condition (14) holds if

$$|a| < 0.335414. \tag{16}$$

Case 3. $\|y\|_2 = \sqrt{y_1^2 + y_2^2}$. Then we derive

$$\begin{aligned}
\|g(t, y)\|_2 &= |a|\sqrt{\sin^2 t(|y_1 + y_2| + 1)^2 + \cos^2 t(y_1 - y_2)^2} \\
&\leq |a|\sqrt{2\sin^2 t + 2\sin^2 t(y_1 + y_2)^2 + \cos^2 t(y_1 - y_2)^2} \\
&\leq \sqrt{2}|a| + |a|\sqrt{2\sin^2 t(y_1 + y_2)^2 + \cos^2 t(y_1 - y_2)^2} \\
&\leq \sqrt{2}|a| + \sqrt{2}|a| \max\left\{\sqrt{2}|\sin t|, |\cos t|\right\} \|y\|_2 \\
&\leq \sqrt{2}|a| + 2|a|\|y\|_2.
\end{aligned}$$

Hence $g_1 = \sqrt{2}|a|$, $g_2 = 2|a|$, and M is given in Case 3 of Example 3.3. So condition (14) holds if

$$|a| < 0.35553. \tag{17}$$

Consequently, the best estimate from equations (15), (16), and (17) is (15), corresponding to the norm $\|\cdot\|_1$, when there is a π-antiperiodic solution $y(t)$ which is nonconstant.

Related results to Examples 3.3–4.3 are given in Battelli and Fečkan (1996); Fečkan et al. (2007), Franco et al. (2003), but our analysis is different since we focus in these examples on finding optimal norms.

2.6 Extension to Mild Solutions

In this section, we extend the above results of equation (4) by assuming the following condition:

(A2) A is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}_{t\in\mathbb{R}_+}$ in X, as stated by Pazy (1983).

We know from Pazy (1983) that there exist constants $Q \geq 1$ and $\gamma \in \mathbb{R}$ such that

$$\|S(t)\| \leq Qe^{\gamma t}, \quad t \geq 0. \tag{18}$$

Thus, we consider in the above definitions and assumptions $\mathbb{R}_+ = [0,\infty)$ instead of $\mathbb{R}$. Then, we look for a (ω, c)-periodic mild solution $y(t) \in C(\mathbb{R}_+, X)$ of equation (4), i.e., y solving the equation

$$y(t) = S(t)y_0 + \int_0^t S(t-s)g(s, y(s))\,ds \tag{19}$$

for some $y_0 \in X$ and any $t \in \mathbb{R}_+$.

Lemma 5.1: Suppose (C1). If there is $y \in Z$ satisfying equations (3) and (19), then the unique extension $y \in \mathcal{Y}_{\omega,c}$ of $y(t)$ to $\mathbb{R}_+$ satisfies equation (19) on the whole $\mathbb{R}_+$.

Proof: The unique extension $y(t)$ is given in the proof of Lemma 2.2. Setting

$$z(t) = S(t)y_0 + \int_0^t S(t-s)g(s, y(s))\,ds$$

and using

$$S(\omega)y_0 + \int_0^\omega S(t-s)g(\omega, y(s))\,ds = y(\omega) = cy(0) = cy_0,$$

We derive for $t \in \mathbb{R}_+$

$$\begin{aligned}
z(t+\omega) &= S(t+\omega)y_0 + \int_0^{t+\omega} S(t+\omega-s)g(s, y(s))\,ds \\
&= S(t)S(\omega)y_0 + \int_0^\omega S(t+\omega-s)g(s, y(s))\,ds + \int_\omega^{t+\omega} S(t+\omega-s)g(s, y(s))\,ds \\
&= S(t)\left(S(\omega)y_0 + \int_0^\omega S(\omega-s)g(s, y(s))\,ds\right) + \int_0^t S(t-s)g(s+\omega, y(s+\omega))\,ds \\
&= cS(t)y_0 + \int_0^t S(t-s)g(s+\omega, cy(s))\,ds \\
&= cS(t)y_0 + c\int_0^t S(t-s)g(s, y(s))\,ds \\
&= cz(t).
\end{aligned}$$

Consequently, it holds that $z \in \mathcal{Y}_{\omega,c}$. But $z(t) = y(t)$ on $[0, \omega]$, and the (ω, c)-periodic extension of $y(t)$ is unique, so $z(t) = y(t)$ on $\mathbb{R}_+$. This means that $y(t)$ satisfies equation (19) on $\mathbb{R}_+$. The proof is finished.

By the above lemma, to find a (ω, c)-periodic mild solution $y(t) \in C(\mathbb{R}_+, X)$ of equation (4), it is equivalent to search for $y \in Z$ satisfying equations (3) and (19). Then we get

$$(cI - S(\omega))y_0 = \int_0^\omega S(\omega - s)g(s, y(s))\, ds.$$

So, extending (A1) to **(A3)** $c \notin \sigma(S(\omega))$, we get

$$y(t) = S(t)(cI - S(\omega))^{-1} \int_0^\omega S(\omega - s)g(s, y(s))\, ds + \int_0^t S(t - s)g(s, y(s))\, ds$$

$$= \int_0^t S(t-s) \left((cI - S(\omega))^{-1}S(\omega) + I\right) g(s, y(s))\, ds + \int_t^\omega S(\omega+t-s)(cI - S(\omega))^{-1} g(s, y(s))\, ds$$

$$= \int_0^t cS(t - s)(cI - S(\omega))^{-1} g(s, y(s))\, ds + \int_t^\omega S(\omega + t - s)(cI - S(\omega))^{-1} g(s, y(s))\, ds.$$

Giving

$$y(t) = \int_0^\omega G(t, s)g(s, y(s))\, ds \tag{20}$$

for

$$G(t, s) = \begin{cases} cS(t - s)(cI - S(\omega))^{-1} & 0 \le s \le t \le \omega, \\ S(\omega + t - s)(cI - S(\omega))^{-1} & 0 \le t < s \le \omega. \end{cases}$$

Next, using equation (18) for any $f \in Z$, we derive

$$\int_0^\omega G(t, s)f(s)\, ds \le \int_0^\omega \|G(t, s)f(s)\|\, ds$$

$$\le \int_0^t \|cS(t - s)(cI - S(\omega))^{-1} f(s)\|\, ds + \int_t^\omega \|S(\omega + t - s)(cI - S(\omega))^{-1} f(s)\|\, ds$$

$$\le Q\|(cI - S(\omega))^{-1}\|\|f\|_0 \left(|c| \int_0^t e^{\gamma(t-s)}\, ds + \int_t^\omega e^{\gamma(\omega+t-s)}\, ds\right)$$

$$= Q\|(cI - S(\omega))^{-1}\|\|f\|_0 \left(|c|\frac{e^{\gamma t} - 1}{\gamma} + \frac{e^{\gamma\omega} - e^{\gamma t}}{\gamma}\right)$$

$$\le U\|f\|_0.$$

Hence, we arrive at

$$\int_0^\omega G(\cdot, s)f(s)\, ds \le U\|f\|_0$$

for any $f \in Z$, where

$$U = \begin{cases} Q\left(\frac{e^{\gamma\omega}-1}{\gamma}\right)\|(cI - S(\omega))^{-1}\| \max\{|c|, 1\} & \gamma \ne 0, \\ Q\omega\|(cI - S(\omega))^{-1}\| \max\{|c|, 1\} & \gamma = 0. \end{cases} \tag{21}$$

Now we can extend Theorem 3.1 as follows:

Theorem 5.2: Suppose (A2), (A3), and consider equation (4) under conditions (C1) and (C2). If

$$LU < 1, \tag{22}$$

then equation (4) has a unique (ω, c)-periodic mild solution y satisfying

$$\|y\|_0 \leq \frac{U\|g(\cdot, 0)\|_0}{1 - LU}. \tag{23}$$

where U is given by equation (21).

Remark 5.3: Note that under (A2) and (C2), there is a unique mild solution of equation (4) on $\mathbb{R}_+$ for any $y_0 \in Y$, depending continuously on y_0 (Pazy, 1983).

If we consider instead of (A2) the following assumption

(A4) A is an infinitesimal generator of a strongly continuous group of bounded linear operators $\{S(t)\}_{t\in\mathbb{R}}$ in X (Pazy, 1983),

then we replace $\mathbb{R}_+$ with $\mathbb{R}$ in the above results.

Example 5.4: We consider a nonlinear heat equation with a forcing term

$$y_t - y_{xx} + \frac{y^3}{2(y^2+1)} = a \sin t, \quad 0 \neq a \in \mathbb{R}, \quad y(0,t) = y(\pi, t) = 0, \quad t \geq 0, \quad x \in [0, \pi]. \tag{24}$$

Now, we have $c = -1$, $\omega = \pi$, and $X = L^2(0, \pi)$ with the norm

$$\|y\| = \sqrt{\int_0^\pi y(t)^2\, dx}$$

and

$$Ay = y_{xx}, \quad D(A) = \{y \in X \mid y', y'' \in X,\, y(0) = y(\pi) = 0\}.$$

Clearly, condition (C1) holds. It is well-known that the sequence $\left\{\frac{\sqrt{2}}{\pi}\sin(kx)\right\}_{k\in\mathbb{N}}$ is an orthonormal basis of X. If

$$y_0(x) = \sum_{k\in\mathbb{N}} y_{0k}\frac{\sqrt{2}}{\pi}\sin(kx) \in X, \quad \|y_0\| = \sqrt{\sum_{k\in\mathbb{N}} y_{0k}^2},$$

then

$$S(t)y_0 = \sum_{k\in\mathbb{N}} e^{-k^2t} y_{0k}\frac{\sqrt{2}}{\pi}\sin(kx) \quad \Rightarrow \quad \|S(t)y_0\| = \sqrt{\sum_{k\in\mathbb{N}} e^{-2k^2t}y_{0k}^2} \leq e^{-t}\|y_0\|.$$

Thus, $Q = 1$ and $\gamma = -1$. Moreover, $\sigma(S(\pi)) = \{e^{-\pi k^2}\}_{k\in\mathbb{N}}$, so (A3) is verified. Next, we derive

$$(-I - S(\pi))^{-1} y_0 = -\sum_{k\in\mathbb{N}} \frac{1}{1+e^{-\pi k^2}} y_{0k} \frac{\sqrt{2}}{\pi} \sin(kx) \quad \Rightarrow \quad \|(-I - S(\pi))^{-1}\| = 1.$$

Hence, $U = 1 - e^{-\pi}$ (see equation (21)). Now, the function $y \to \frac{y^3}{2(y^2+1)}$ has a Lipschitz constant $\frac{9}{16}$, so we have $L = \frac{9}{16}$ in (C2). Thus,

$$LU = \frac{9(1-e^{-\pi})}{16} \approx 0.538192 < 1,$$

and condition (22) is verified. By Theorem 5.2, equation (24) has a unique π-antiperiodic mild solution for any $0 \neq a \in \mathbb{R}$ satisfying

$$\max_{t\in[0,\pi]} \|y(\cdot,t)\| \leq 16(e^{\pi}-1)\sqrt{\pi}\left(\frac{9+7e^{\pi}}{9}\right)|a| \approx 3.67222|a|.$$

Example 5.5: We consider a nonlinear Schrödinger equation with a forcing term

$$\frac{1}{i} y_t - y_{xx} + \frac{|y|^2}{5(|y|^2+1)} = a(1+\sin^2 x)e^{t/4i}, \quad 0 \neq a \in \mathbb{C}, \quad y(x,t) = y(x+2\pi,t) = 0, \quad t \geq 0, \quad x \in \mathbb{R}. \tag{26}$$

Now we have $c = e^{\frac{\pi}{4}i} = 1 + \frac{\sqrt{i}}{2}$, $\omega = \pi$, and $X = L^2(0,2\pi)$ with the norm

$$\|y\| = \sqrt{\int_0^{2\pi} |y(t)|^2\, dt}$$

and

$$Ay = iy_{xx}, \quad D(A) = \{y \in X \mid y', y'' \in X,\, y(0) = y(\pi),\, y'(0) = y'(\pi)\}.$$

Clearly, condition (C1) holds. It is well-known that the sequence $\left\{\frac{1}{\sqrt{2\pi}} e^{kxi}\right\}_{k\in\mathbb{Z}}$ is an orthonormal basis of X. If

$$y_0(x) = \sum_{k\in\mathbb{Z}} y_{0k} \frac{1}{\sqrt{2\pi}} e^{kxi} \in X, \quad \|y_0\| = \sqrt{\sum_{k\in\mathbb{Z}} y_{0k}^2},$$

then

$$S(t)y_0 = \sum_{k\in\mathbb{Z}} e^{-k^2 t} i y_{0k} \frac{1}{\sqrt{2\pi}} e^{kxi} \quad \Rightarrow \quad \|S(t)y_0\| = \sqrt{\sum_{k\in\mathbb{Z}} y_{0k}^2} = \|y_0\|.$$

Hence, $Q = 1$ and $\gamma = 0$. Note that now (A4) holds. Moreover, $\sigma(S(\pi)) = \{\pm 1\}$, so (A3) is verified. Next, we derive

$$\left(1 + \frac{i}{\sqrt{2}} I - S(\pi)\right)^{-1} y_0 = \sum_{k\in\mathbb{Z}} \frac{1}{1 + \frac{\sqrt{i}}{2} - e^{-\pi k^2 i}} y_{0k} \frac{1}{\sqrt{2\pi}} e^{kxi}$$

$$\Rightarrow \quad \left(1 + \frac{i}{\sqrt{2}} I - S(\pi)\right)^{-1} = \sqrt{1 + \frac{1}{\sqrt{2}}} \approx 1.30656.$$

Hence, $U = \pi\sqrt{1+\frac{1}{\sqrt{2}}} \approx 4.10469$ (see equation (21)). Now, the function $H(y) = |y|^2 y \frac{5}{(|y|^2+1)}$, where $H : \mathbb{C} \to \mathbb{C}$, has a derivative

$$DH(y)v = (|y|^4 + 2|y|^2)v + \frac{y^2 v^*}{5(|y|^2+1)^2},$$

for v^* denoting the complex conjugate of $v \in \mathbb{C}$. Hence,

$$|DH(y)v| \leq \frac{|y|^4 + 3|y|^2}{5(|y|^2+1)^2}|v| \leq \frac{9}{40}|v|.$$

Thus, H has a Lipschitz constant $\frac{9}{40}$, so we have $L = \frac{9}{40}$ in (C2). Thus,

$$LU = \frac{9\pi}{40}\sqrt{1+\frac{1}{\sqrt{2}}} \approx 0.923555 < 1,$$

and condition (22) is verified. By Theorem 5.2, equation (26) has a unique $(\pi, 1+\frac{\sqrt{i}}{2})$-periodic mild solution for any $0 \neq a \in \mathbb{C}$ satisfying

$$\max_{t\in[0,2\pi]} \|y(\cdot,t)\| \leq \frac{20\sqrt{38}}{(2+\sqrt{2})\pi^{3/2}} \left(80 - 9\sqrt{2}(2+\sqrt{2})\right)|a| \approx 207.421|a|.$$

Finally, we extend Theorem 4.1 as follows.

Theorem 5.6: Assume (A2), (A3), (C1), (C2), (C3) along with (A5) that $S(t)$ is compact for any $t > 0$. If

$$Q\|(cI - S(\omega))^{-1}\|e^{\gamma\omega}(e^{Qg_2\omega} - 1) < 1, \quad (27)$$

then (4) has a (ω, c)-periodic mild solution.

Proof: Recalling Remark 5.3, we denote by $y(y_0,t)$, $t \in \mathbb{R}_+$, the unique mild solution of (4) and introduce a mapping

$$P(y_0) = (cI - S(\omega))^{-1}\int_0^\omega S(\omega - s)g(s, y(y_0,s))\,ds. \tag{28}$$

Note that $y_0 = P(y_0)$ is equivalent to $y(y_0,\omega) = cy_0$, so fixed points of P determine (ω, c)-periodic mild solutions of (4).

Next, equations (18), (19), and (C3) imply

$$\|y(y_0,t)\| \leq Qe^{\gamma t}\|y_0\| + Q\int_0^t e^{\gamma(t-s)}(g_1 + g_2\|y(y_0,s)\|)\,ds$$

$$= Qe^{\gamma t}\|y_0\| + Qg_1e^{\gamma t}\int_0^t e^{-\gamma s}\,ds + Qg_2e^{\gamma t}\int_0^t \|y(y_0,s)\|e^{-\gamma s}\,ds.$$

$$\leq Qe^{\gamma t}\|y_0\| + Qg_1e^{\gamma t}\omega e^{|\gamma|\omega} + Qg_2e^{\gamma t}\int_0^t \|y(y_0,s)\|e^{-\gamma s}\,ds.$$

This gives

$$\|y(y_0,t)\|e^{-\gamma t} \leq Q\|y_0\| + Qg_1\omega e^{|\gamma|\omega} + Qg_2\int_0^t \|y(y_0,s)\|e^{-\gamma s}\,ds$$

for $t \in [0, \omega]$, so by the Gronwall inequality, we get

$$\|y(y_0, t)\| \le Q\left(\|y_0\| + g_1 \omega e^{|\gamma|\omega}\right) e^{(Qg_2+\gamma)t} \quad \forall t \in [0, \omega].$$

Then, equation (28) has the estimate

$$\|P(y_0)\| \le \|(cI - S(\omega))^{-1}\| \int_0^\omega \|S(\omega - s)\|(g_1 + g_2\|y(y_0, s)\|)\, ds$$

$$\le Q\|(cI - S(\omega))^{-1}\| \int_0^\omega e^{\gamma(\omega-s)} \left(g_1 + g_2 Q\left(\|y_0\| + g_1\omega e^{|\gamma|\omega}\right) e^{(Qg_2+\gamma)s}\right) ds$$

$$\le Q\|(cI - S(\omega))^{-1}\| \left(g_1 e^{|\gamma|\omega}\omega + g_2 e^{\gamma\omega} Q\left(\|y_0\| + g_1\omega e^{|\gamma|\omega}\right) \int_0^\omega e^{Qg_2 s}\, ds\right)$$

$$= Q\|(cI - S(\omega))^{-1}\| \left(g_1 e^{|\gamma|\omega}\omega + e^{\gamma\omega}\left(\|y_0\| + g_1\omega e^{|\gamma|\omega}\right)(e^{Qg_2\omega} - 1)\right)$$

$$= Q\|(cI - S(\omega))^{-1}\| e^{\gamma\omega}(e^{Qg_2\omega} - 1)\|y_0\| + Q\|(cI - S(\omega))^{-1}\| \left(g_1 e^{|\gamma|\omega}\omega + e^{\gamma\omega} g_1\omega e^{|\gamma|\omega}(e^{Qg_2\omega} - 1)\right).$$

Hence, by (27) and taking $y_0 \in X$ such that

$$\|y_0\| \le \Xi = \frac{Q\|(cI - S(\omega))^{-1}\| \left(g_1 e^{|\gamma|\omega}\omega + e^{\gamma\omega} Q g_1\omega e^{|\gamma|\omega}(e^{Qg_2\omega} - 1)\right)}{1 - Q\|(cI - S(\omega))^{-1}\| e^{\gamma\omega}(e^{Qg_2\omega} - 1)},$$

we get $\|P(y_0)\| \le \Xi$, i.e., $P : B(\Xi) \to B(\Xi)$. We already know that P is continuous. Now, we show that P is also compact. For any $n \in \mathbb{N}, n > \frac{1}{\omega}$, we set

$$P_n(y_0) = (cI - S(\omega))^{-1} \int_0^{\omega - n^{-1}} S(\omega - s) g(s, y(y_0, s))\, ds.$$

Then

$$P_n(y_0) = S(n^{-1})(cI - S(\omega))^{-1} \int_0^{\omega - n^{-1}} S(\omega - n^{-1} - s) g(s, y(y_0, s))\, ds.$$

Since

$$(cI - S(\omega))^{-1} \int_0^{\omega - n^{-1}} S(\omega - n^{-1} - s) g(s, y(y_0, s))\, ds$$
$$\le Q\|(cI - S(\omega))^{-1}\| \int_0^{\omega - n^{-1}} e^{\gamma(\omega - n^{-1} - s)} \left(g_1 + g_2 Q\left(\Xi + g_1\omega e^{|\gamma|\omega}\right) e^{(Qg_2+\gamma)s}\right) ds$$

$$\le Q\|(cI - S(\omega))^{-1}\| \omega e^{|\gamma|\omega} \left(g_1 + g_2 Q\left(\Xi + g_1\omega e^{|\gamma|\omega}\right) e^{(Qg_2+\gamma)\omega}\right)$$

for any $y_0 \in B(\Xi)$, by (A5), $P_n(B(\Xi))$ is precompact. Furthermore, we derive

$$\|P(y_0) - P_n(y_0)\| \le Q\|(cI - S(\omega))^{-1}\| \int_{\omega - n^{-1}}^{\omega} e^{\gamma(\omega - s)} \left(g_1 + g_2 Q\left(\Xi + g_1\omega e^{|\gamma|\omega}\right) e^{(Qg_2+\gamma)s}\right) ds$$

$$\le Q\|(cI - S(\omega))^{-1}\|e^{|\gamma|\omega}\left(g_1 + g_2 Q\left(\Xi + g_1\omega e^{|\gamma|\omega}\right)e^{(Qg_2+\gamma)\omega}\right)n^{-1},$$

hence $P_n \rightrightarrows P$ uniformly on $B(\Xi)$. This gives the precompactness of $P(\Xi)$. Summarizing, we can apply the Schauder fixed point theorem to P, which finishes the proof.

For illustration of Theorem 5.6, we consider a parametrized version of Example 5.4 in the form:

Example 5.7: We consider a nonlinear heat equation with a forcing term

$$y_t - y_{xx} + \eta y^3\left(y^2+1\right) = a\sin t, \quad 0 \neq a \in \mathbb{R}, \quad y(0,t) = y(\pi,t) = 0, \quad t \geq 0, \quad x \in [0,\pi],$$

for a parameter $\eta > 0$. Then, we have $L = \frac{9\eta}{8}$ in (C2). Thus,

$$LU = \frac{9(1-e^{-\pi})}{8\eta}.$$

and (22) is verified for

$$0 < \eta < \frac{8}{9}\left(1 - e^{-\pi}\right) \approx 0.929036. \tag{30}$$

By Theorem 5.2, (29) has a unique π-antiperiodic mild solution for any $0 \neq a \in \mathbb{R}$ and η satisfying (30).

On the other hand, for any $y \in X = L^2(0,\pi)$, we have

$$a\sin t - \frac{\eta y^3}{y^2+1} \le |a|\|\sin t\| + \eta\sqrt{\int_0^\pi y^6(t)\left(y^2(t)+1\right)^2 dt}$$

$$\le |a|\sqrt{\frac{\pi}{2}} + \eta\sqrt{\int_0^\pi y^2(t)dt} = \sqrt{\frac{\pi}{2}} + \eta\|y\|.$$

Thus, (C3) is verified for $g_1 = |a|\sqrt{\frac{\pi}{2}}$ and $g_2 = \eta$. Then, (27) has the form

$$e^{-\pi}\left(e^{\eta\pi} - 1\right) < 1,$$

i.e.,

$$0 < \eta < \frac{\ln(1+e^{\pi})}{\pi} \approx 1.01347. \tag{31}$$

Next, since $e^{-k^2 t} \to 0$ as $k \to \infty$ for any $t > 0$, by (25), the compactness of $S(t)$, $t > 0$, is clear and well-known. By Theorem 5.6, (29) has a π-antiperiodic mild solution for any $0 \neq a \in \mathbb{R}$ and η satisfying (31), and it is unique when (30) holds.

Chapter 3

(ω, c)-Pseudo periodic functions, first order Cauchy problem and Lasota–Wazewska model with ergodic and unbounded oscillating production of red cells

Sommaire

This work investigates the existence and uniqueness of (ω, c)-pseudo periodic solutions for first-order differential equations, with a focus on the Lasota–Wazewska model that describes the dynamics of red blood cells. The model is analyzed under the influence of an **ergodic and unbounded oscillatory production**, which reflects complex biological rhythms. Using tools from functional analysis and fixed point theorems, we demonstrate that, under suitable conditions, the system admits a unique (ω, c)-pseudo periodic solution. This extends the classical framework to more realistic, non-periodic settings.

Applications to Abstract Integral and Differential Equations in Banach Spaces

Consider the integral equation (see [21]):

$$u(t) = \int_{-\infty}^{t} R(t, s) f(s, u(s))\, ds, \tag{3.1}$$

where f and R satisfy the following hypotheses.

(H1)

$$f(t, x) = g(t, x) + h(t, x), \quad \text{where} \quad g(t + \omega, cx) = cg(t, x) \quad \text{and} \quad h \in \mathcal{A}_{A_0,c}(X, X),$$

and satisfies the condition

$$f(t, x) - f(t, y) \leq L_f \|x - y\|, \quad t \in \mathbb{R}, \quad x, y \in X,$$

where $L_f > 0$.

(H2)

$$h_t(x) := c^\wedge(-t) h(t, c^\wedge(t)x) \quad \text{is uniformly continuous for} \quad x \in \text{any bounded set of } X \quad \text{uniformly in} \quad t \in \mathbb{R}$$

and

$$\lim_{T\to\infty} \frac{1}{2T} \int_{T-T}^{T} c^\wedge(-t) h_t, c^\wedge(t)x \, dt = 0, \quad \text{for} \quad x \in \text{any bounded subset of } X.$$

(H3) The kernel R satisfies the inequality

$$R(t,s) \leq Mk(t-s), \quad t \geq s, \quad M > 0,$$

where $k \sim (t) = c^\wedge(-t)k(t) \in L^1([0,\infty))$.

(H4) $R(t,s)$ is bi-periodic in the sense of

$$R(t+\omega, s+\omega) = R(t,s), \quad t \geq s. \tag{3.2}$$

Note that, for an arbitrary periodic function a, the kernel defined by the following relation

$$R(t,s) := \exp\left(\int_s^t a(r)\,dr\right)$$

satisfies hypothesis (H4).

Theorem 3.0.1. *Assume that (H1)–(H4) hold. Then, if $L_f Mk \sim 1 < 1$, the integral equation*

$$(3.1) \quad u(t) = \int_{-\infty}^{t} R(t,s) f(s,u(s))\,ds$$

has a unique (ω, c)-pseudoperiodic solution.

Proof: We define $G : \mathrm{PP}_{\omega,c}(X) \to \mathrm{PP}_{\omega,c}(X)$ by

$$(Gu)(t) = \int_{-\infty}^{t} R(t,s) f(s,u(s))\,ds$$

for $u \in \mathrm{PP}_{\omega,c}(X)$ and $t \in \mathbb{R}$.

First, we prove that the operator G is well-defined. Indeed, let $\varphi(\cdot) = f(\cdot, u(\cdot))$. By Theorem 2.25, we have that $\varphi \in \mathrm{PP}_{\omega,c}(X)$. Then,

$$\|Gu\|_{\mathrm{p}\omega c} \leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} c^\wedge(-t) R(t,s)\varphi(s)\,ds \leq M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} c^\wedge(-(t-s))k(t-s)c^\wedge(-s)\varphi(s)\,ds$$

$$\leq M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \tilde{k}(t-s)c^\wedge(-s)\varphi(s)\,ds \leq M\|\varphi\|_{\mathrm{p}\omega c} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \tilde{k}(t-s)\,ds < \|\varphi\|_{\mathrm{p}\omega c}\tilde{k}_1 < \infty.$$

Now, since $\varphi \in \mathrm{PP}_{\omega,c}(X)$, there exist functions $\varphi_1 \in P_{\omega,c}(\mathbb{R}, X)$ and $\varphi_2 \in \mathcal{A}_{A_0,c}(X)$ such that

$$\varphi = \varphi_1 + \varphi_2.$$

Then we can split $G = G_1 + G_2$, where

$$(G_1u)(t) := \int_{-\infty}^{t} R(t,s)\varphi_1(s)\,ds, \quad (G_2u)(t) := \int_{-\infty}^{t} R(t,s)\varphi_2(s)\,ds.$$

First, we prove that $G_1 \in P_{\omega,c}(\mathbb{R}, X)$. By (H4), we have that

$$(G_1u)(t+\omega) := \int_{t+\omega}^{-\infty} R(t+\omega, s)\varphi_1(s)\,ds = \int_{t}^{-\infty} R(t,s)\varphi_1(s)\,ds = c\int_{-\infty}^{t} R(t,s)\varphi_1(s)\,ds = c(G_1u)(t).$$

It follows that $G_1 \in P_{\omega,c}(\mathbb{R}, X)$.

Next, we prove that $G_2 \in \mathcal{A}_{A_0,c}(X)$, that is,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{T-T}^{T} c^\wedge(-t)(G_2u)(t)\, dt = 0.$$

By (H3), we have that

$$\begin{aligned} c^\wedge(-t)(G_2u)(t) &\le \int_{-\infty}^{t} |c|^\wedge(-(t-s))R(t,s)|c|^\wedge(-s)\varphi_2(s)\, ds \\ &\le M \int_{-\infty}^{t} |c|^\wedge(-(t-s))k(t-s)|c|^\wedge(-s)\varphi_2(s)\, ds \\ &= M \int_{-\infty}^{t} k^\sim(t-s)c^\wedge(-s)\varphi_2(s)\, ds. \end{aligned}$$

Since $\varphi_2 \in AA_0, c(X)$, the conclusion follows from Convolution Theorem 2.21. Therefore, $G(PP_\omega c(X)) \subset PP_\omega c(X)$. Now, if $u, v \in PP_\omega c(X)$, we have

$$G(u) - G(v)_{\omega c} = \sup_{t\in\mathbb{R}} \left(\sup |c| \wedge (-t) \int_{-\infty}^{t} R(t,s) f_s, u(s) - f_s, v(s)\, ds \right).$$

$$\le ML_f \int_0^\infty k \sim (s)\, ds \le ML_f u - v_{\omega c} k \sim 1.$$

It follows from the Banach Fixed Point Theorem that there exists a unique $u \in PP_\omega c(X)$ such that $Gu = u$, that is, $u(t) = \int_{-\infty}^{t} R(t,s) f(s, u(s))\, ds$.

The previous results can be applied to obtain (ω, c)-pseudo periodic solutions to the semilinear evolution equation

$$u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}.$$

We assume the following condition.

(H_5) The operator A generates an exponentially stable C_0-semigroup $(T(t))_{t\ge 0}$, that is, there exist constants $M > 0$ and $\alpha > 0$ such that $T(t) \le Me^{-\alpha t}$ for each $t \ge 0$ and $c > e^{-\alpha}$.

Thus, we have the following theorem.

Theorem 3.0.2. *Assume that* (H1) *and* (H5) *hold. Then Eq. (3.3) has a unique (ω, c)-pseudo periodic solution whenever*

$$ML_f < |\widetilde{k}|^{-1},$$

where $\widetilde{k}(t) = c(-t)e^{-\alpha t}$.

3.0.1 Lasota–Wazewska Model with Unbounded Oscillating and Ergodic Production of Red Cells

The theory presented above can be extended to the semilinear abstract problem with delay:

$$\begin{cases} y(t) = Ay(t) + f(t, y(t-\tau)), & t \geq 0, \\ y(t) = \varphi(t), & t \in [-\tau, 0], \end{cases}$$

where $\tau > 0$ and for which a mild solution is a solution of the integral equation

$$y(t) = T(t)y(0) + \int_0^t T(t-s)f(s, y(s-\tau))\,ds, \quad t \geq 0.$$

Here, we need to know the history φ. Note that $y(t-\tau) = \varphi(t-\tau)$ for $t \in [0, \tau]$, and if y is (ω, c)-pseudo periodic, then $y(t-\tau)$ also is. As an example, we study the important Lasota–Wazewska model with (ω, c)-pseudo periodic variable coefficients. The Lasota–Wazewska model is an autonomous differential equation of the form

$$y(t) = -\delta y(t) + he^{-\gamma y(t-\tau)}, \quad t \geq 0. \tag{3.1}$$

Wazewska–Czyzewska and Lasota [**?**] proposed this model to describe the survival of red blood cells in the blood of an animal. In this equation, $y(t)$ describes the number of red blood cells in the blood at time t, $\delta > 0$ is the probability of death of a red blood cell, h and γ are positive constants related to the production of red blood cells per unit of time, and τ is the time required to produce a red blood cell.

In this section, we study the following model:

$$y(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, \quad t \geq 0, \tag{3.2}$$

where $\tau > 0$, $h(t)$ and $a(t)$ are continuous and positive functions. Equation (4.2) models several situations in real life and the references therein. We are looking for positive (ω, c)-pseudo periodic solutions for certain $\omega > 0, c > 0$. Let

$$f(t, y) = h(t)e^{-a(t)y}$$

and assume the following conditions:

(a) $\tau \leq \omega$;

(b) h is (ω, c)-pseudo periodic;

(c) a is $(\omega, \frac{1}{c})$-pseudo periodic;

(d) $c > e^{-\delta\omega}$;

(e) $ah_\infty < \delta$.

Remember that if $y(\cdot) \in PP_{\omega c}(X)$, then $y(\cdot - \tau) \in PP_{\omega c}(X)$. By (d) and (e), we have that $f(t, y) = h(t)e^{-a(t)y}$ satisfies the hypotheses of Theorem 3.2, since

$$|f(t, y_1) - f(t, y_2)| \leq a(t)h(t)|y_1 - y_2|$$

for $y_1, y_2 > 0$, and its (ω, c)-pseudo periodic part g satisfies

$$g(t + \omega, cy) = cg(t, y).$$

By the variation of constant formula:

$$y(t) = e^{-\delta t}y(0) + \int_0^t e^{-\delta(t-s)} f(s, y(s-\tau))\, ds,$$

we can deduce that $y(0) > 0$ implies $y(t) > 0$. Note that condition (d) is necessary for positive c-periodic solutions y. In fact, (4.5) and $h(t) > 0$ imply $y(t) > e^{-\delta t}y(0)$, which evaluated at $t = \omega$ implies (d) since $[c - e^{-\delta\omega}]y(0) > 0$.

Moreover, taking $y(0) = \int_{-\infty}^0 e^{\delta s} f(s, y(s-\tau))\, ds$, which is well-defined, we have that y satisfies

$$y(t) = -\int_{-\infty}^t e^{-\delta(t-s)} f(s, y(s-\tau))\, ds.$$

Then, by Theorem 3.2, we have that this equation has a unique solution y^*, which belongs to $PP_{\omega c}(X)$. Hence, y^* is also a solution of type $PP_{\omega c}(X)$ of equation (4.2). Moreover, y^* is exponentially stable. Indeed, for any solution y of (4.2), $z = y - y^*$ satisfies

$$z = -\delta z + f(t, y) - f(t, y^*) = -\delta z + f(t, y^*) + z - f(t, y^*).$$

Note that

$$f(t, y^*) + z - f(t, y^*) \leq a(t)h(t)|z|.$$

Then, taking $ah_\infty < \delta$, z verifies that

$$z(t) \leq e^{-\alpha(t-t_0)} \sup_{t_0-\tau\leq s\leq t_0} |z(s)|$$

for $t \geq t_0 \geq 0$ and $0 < \alpha < \delta - ah_\infty$.

We have proved the following theorem.

Theorem 3.0.3. *Assume that conditions (a) to (e) hold. Then the Lasota–Wazewska model has a unique (ω, c)-pseudo periodic solution.*

Conclusion

To conclude, our work focused on the study of (ω, c)-periodic and (ω, c)-pseudo periodic functions, which generalize the classical notion of periodicity to settings where exact repetition is replaced by more flexible recurrence conditions. These classes of functions are crucial in modeling phenomena where behavior repeats in a controlled, yet not strictly periodic, manner—such as in biology, control theory, and signal processing.

We began by introducing the basic definitions and illustrating the motivations for considering such generalized periodic functions. We then explored their key properties, including linearity, translation invariance, and their role in the stability analysis of solutions to differential and integral equations.

In the second part, we established the existence and uniqueness results of (ω, c)-periodic and (ω, c)-pseudo periodic solutions using fixed point theorems such as Banach and Schauder. These theoretical results show the richness of the functional framework and provide a solid foundation for future analytical and numerical applications.

Finally, this study opens the door to further research, including the extension to functions with values in Banach spaces, the study of pseudo almost periodic analogs, or the analysis of delay differential equations within this framework.

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