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MEMOIRE

En vue de l'obtention du diplôme de MASTER en :

Géométrie différentielle

Intitulé

Efficient algorithms for obtaining explicit exact solutions to some PDEs in various scientific fields

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ملخص

في هذا العمل، بحثنا في حلول معادلتين تطوريتين غير خطيتين مهمتين ونظامين مقترنين. واستُخدمت أساليب تحليلية متنوعة لاستخلاص ثلاث فئات من الحلول الدقيقة - الزائدية، والمثلثية، والنسبية - بالإضافة إلى الحلول البصرية. وقد سهّل استخدام برامج حاسوبية مثل *Mathematica 11* بشكل كبير إجراء عمليات حسابية جبرية معقدة، ومكّن من تصور أسطح الحلول، مما ساعد في تفسير كل من السلوك الديناميكي والبنية الهندسية للحلول.

الأساليب المستخدمة في هذه الرسالة قياسية، ومباشرة، ومناسبة تمامًا للتطبيق الحاسوبي. وبشكل عام، تُعتبر هذه التقنيات أدوات رياضية فعالة وقوية للغاية، ذات قابلية تطبيق واسعة في حل مجموعة واسعة من المعادلات التفاضلية الجزئية غير الخطية. علاوة على ذلك، يمكن توسيع نطاقها لتشمل مجالات مختلفة، بما في ذلك الفيزياء الرياضية، والهندسة، وغيرها من المجالات العلمية غير الخطية.

نأمل أن تُسهم الحلول التي تم الحصول عليها في هذا العمل في فهم أعمق لبعض الظواهر الفيزيائية غير الخطية.

Abstract

In this work, we investigated solutions to two important nonlinear evolution equations and two coupled systems. Various analytical methods were employed to derive three classes of exact solutions—hyperbolic, trigonometric, and rational—as well as optical solutions. The use of computational software such as *Mathematica 11* significantly facilitated the execution of complex algebraic manipulations and enabled the visualization of solution surfaces, aiding in the interpretation of both the dynamic behavior and geometric structure of the solutions.

The methods utilized in this thesis are standard, direct, and well-suited for computer implementation. Overall, these techniques are recognized as highly effective and powerful mathematical tools, with broad applicability in solving a wide range of nonlinear partial differential equations. Moreover, they can be extended to various domains, including mathematical physics, engineering, and other nonlinear scientific fields.

It is our hope that the solutions obtained in this work will contribute to a deeper understanding of certain nonlinear physical phenomena.

Résumé

Dans ce travail, nous avons étudié les solutions de deux importantes équations d'évolution non linéaires ainsi que de deux systèmes couplés. Diverses méthodes analytiques ont été utilisées pour obtenir trois classes de solutions exactes — hyperboliques, trigonométriques et rationnelles — ainsi que des solutions optiques. L'utilisation de logiciels de calcul tels que *Mathematica 11* a grandement facilité l'exécution de manipulations algébriques complexes et permis la visualisation des surfaces de solutions, contribuant ainsi à l'interprétation du comportement dynamique et de la structure géométrique des solutions.

Les méthodes employées dans cette thèse sont standards, directes et bien adaptées à une mise en œuvre informatique. De manière générale, ces techniques sont reconnues comme des outils mathématiques puissants et efficaces, avec une large applicabilité à la résolution d’un grand nombre d’équations aux dérivées partielles non linéaires. Elles peuvent également être étendues à divers domaines, notamment la physique mathématique, l’ingénierie et d’autres sciences non linéaires.

Nous espérons que les solutions obtenues dans ce travail contribueront à une meilleure compréhension de certains phénomènes physiques non linéaires.

Acknowledgement

I thank Allah above all, for to Him alone belongs all praise. I would like to express our deep gratitude to our supervisor, **Dr. Djilali Medjahed**, who allowed me to considerably enrich my mathematical knowledge and who helped us write and complete my thesis using LaTex, as well as use the Mathematica software to perform operations quickly and also plot curves with precision and clarity. Thank you for your generosity, Professor!

I extend special thanks to **Dr. Guendouz Cheikh** for his valuable advice, his boundless dedication, his numerous encouragements, and his compassionate nature. It is a pleasure and an honor for me to warmly thank him for accepting the presidency of the jury.

I am deeply touched by the honor bestowed upon us by **Dr. Mahdi Fatima zohra** by agreeing to serve on the jury as an examiner. I would like to express my deepest gratitude for his interest in our work.

I would like to take this opportunity to thank all my Master's and Bachelor's professors.

I would like to extend our greetings to all the members of my class.

To my family.

To all my friends..

Univ-Relizane Chaimaa LAHOUACHE

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Dedicate

This modest work is dedicated to my family,

My supervisor Dr. DJILALI Medjahed

My Colleagues of University of Relizane

Chaimaa LAHOUACHE

Preface

In this thesis, we focus on the study of nonlinear partial differential systems and equations (in particular, nonlinear evolution equations). This is done by seeking exact solutions-especially traveling wave solutions-obtained analytically using new methods.

In the first chapter, we included the definitions and concepts of evolution equations and the description of the modified exponential function method, provided basic ideas of the generalized exponential rational function method, described the improved generalized Riccati equation mapping method, cited the basic ideas the Modified Exponential Function Method, as well as the basic ideas of the Extended (G'/G)-expansion method.

In the second chapter, we studied the (1+1)-dimensional improved Boussinesq equation. It is one of the most popular equations in soliton physics and has appeared in nonlinear model equations governing two-dimensional irrotational flows of an inviscid fluid in a uniform rectangular channel. It can also be encountered in other physical applications such as nonlinear lattice waves, acoustic waves, ion-acoustic waves in plasma, and vibrations in a nonlinear string. Using the generalized exponential rational function method, we obtained a variety of exact solutions.

In the third chapter, we focused on obtaining exact solutions for the coupled (1+1)-dimensional Long-Short Wave Interaction equation, by applying the generalized Riccati mapping method. This system of equations admits a variety of complex solutions (so-called optical solitons). It is worth noting that this system is a mathematical model that describes the nonlinear interaction between a long wave and a short wave. Such interactions are common in fluid dynamics, plasma physics, nonlinear optics, and other fields where waves of different scales coexist and influence each other.

The fourth chapter is devoted to the study of the coupled Klein-Gordon system. By using the modified exponential function method, we were able to find new explicit solutions to this system. This system of partial differential equations (PDEs) generalizes the standard Klein-Gordon equation to include multiple interacting fields. These equations frequently appear in theoretical physics, particularly in quantum field theory, nonlinear optics, and condensed matter physics.

Finally, in the fifth chapter, we addressed the (1+1)-dimensional integro-differential Ito equation. We implemented the extended (G'/G)-expansion method to derive new analytical solutions. It is worth noting that the Ito equation models phenomena arising in shallow water waves.

Contents

Chapter 1

Introduction and Preliminary

1.1 Introduction

1.1.1 Nonlinear Partial Differential Equations

In mathematics and physics, a nonlinear partial differential equation (PDE) is a partial differential equation with nonlinear terms. These equations describe many different physical systems, ranging from gravitation to fluid dynamics, and have been used in mathematics to solve problems such as the Poincaré conjecture and the Calabi conjecture. They are challenging to study: there are almost no general techniques that work for all such equations, and typically each individual equation must be studied as a separate problem.

1.1.2 Evolution Equation

An equation that can be interpreted as the differential law of the development (evolution) over time of a system. The term has no exact definition, and its meaning depends not only on the equation itself but also on the formulation of the problem for which it is used. An evolution equation typically allows the construction of a solution from a prescribed initial condition, which can be interpreted as a description of the initial state of the system. The class of evolution equations includes, first and foremost, ordinary differential equations and systems of the form:

$$u' = f(t,u), \quad u'' = f(t,u,u'), \tag{1.1}$$

etc., where $u(t)$ can naturally be considered as the solution to the Cauchy problem; these equations describe the evolution of systems with finite degrees of freedom. Accounting for secondary effects leads to integro-differential Volterra equations or differential equations with delayed arguments. The description of processes occurring in continuous media reduces to partial differential equations of hyperbolic, parabolic, or elliptic types; here, alongside the Cauchy problem, one can also pose a mixed (initial boundary value) problem. If the solution $u(x,t)$ of such an equation is considered as an element of a certain function space in x that depends on a parameter t, then we arrive at abstract differential equations of the form (1.1.1). All these equations, as well as their corresponding difference equations, generally belong to the class of evolution equations.

1.1.3 Study Methods for Nonlinear Partial Differential Equations

A fundamental question for any PDE is the existence and uniqueness of a solution for given boundary conditions. For nonlinear equations, these questions are generally very difficult: for example, the most challenging part of Yau's solution to the Calabi conjecture was the proof of the existence of a solution to the Monge-Ampère equation.

Linear Approximation: Solutions near a known solution can sometimes be studied by linearizing the PDE around the solution. This corresponds to studying the tangent space of a point in the moduli space of all solutions.

Exact Solutions: It is often possible to explicitly write down certain special solutions in terms of elementary functions (though it is rarely possible to describe all solutions this way). One way to find such explicit solutions is to reduce the equations to lower-dimensional equations, preferably ordinary differential equations, which can often be solved exactly. This can sometimes be achieved using separation of variables or by seeking highly symmetric solutions. Some equations have multiple distinct exact solutions.

Numerical Solutions: Numerical solutions on a computer are almost the only method that can be used to obtain information about arbitrary PDE systems. Much work has been done, but there is still much to be done to solve certain systems numerically, particularly for Navier-Stokes and other equations related to weather prediction.

1.1.4 PDEs and Computers

The study of partial differential equations (PDEs) dates back to the 18th century, following analytical investigations of a wide range of models (works by Euler, Cauchy, d'Alembert, Hamilton, Jacobi, Lagrange, Laplace, Monge, and many others). Since the mid-19th century (works by Riemann, PoincarÃ©, Hilbert, and others), PDEs have become an essential tool for studying other branches of mathematics. The most significant results in determining explicit solutions of nonlinear PDEs were obtained by S. Lie. Many analytical methods rely on Lie symmetries (or continuous symmetry transformation groups). Today, these transformations can be performed using computer algebra systems (e.g., Maple and Mathematica). Currently, PDE theory plays a central role in the general advancement of mathematics, as it helps describe the evolution of many phenomena in various fields of science, engineering, and many other applications. Since the 20th century, the study of nonlinear PDEs has become an independent field developing in many research directions. One such direction is the symbolic and numerical computation of solutions to nonlinear PDEs, which is considered in this thesis. It is worth noting that the main ideas for practical computations of PDE solutions were first indicated by H. PoincarÃ© in 1890. However, the techniques for solving such problems required technology that was either unavailable or limited at the time. In modern mathematics, there are computers, supercomputers, and computer algebra systems (such as Maple and Mathematica) that can assist in performing various mathematical operations for which humans have limited capacity, and where symbolic and numerical computations play a central role in scientific progress. It is known that there are different analytical solution methods for special nonlinear PDEs; however, in the general case, there is no unified theory of nonlinear PDEs. There is no single method that can be applied to all types of nonlinear PDEs. Although ”nonlinearity” makes each equation or problem unique, new processes must be discovered to solve at least one class of nonlinear PDEs. Moreover, the functions and data in nonlinear PDE problems are frequently defined at discrete points. Therefore, numerical approximation methods for nonlinear PDEs must be studied. Scientists typically apply different approaches to study nonlinear partial differential equations.

1.1.5 Traveling Wave Solutions of Nonlinear PDEs

A traveling wave is a wave that advances in a particular direction while maintaining a fixed shape. Additionally, a traveling wave is associated with a constant speed throughout its propagation path. Such waves are observed in many fields of science, such as combustion, which can occur as a result of a chemical reaction. In mathematical biology, apparent pulses in nerve fibers are represented as traveling waves. Furthermore, in conservation laws associated with fluid dynamics problems, shock profiles are characterized as traveling waves. Additionally, structures present in solid mechanics are typically modeled as standing waves. It is therefore important to determine the dynamics of such solutions. A traveling wave solution is obtained when solving a model corresponding to a system. Generally, these models take the form of partial differential equations (PDEs), where the dynamics of the systems are understood when solving the solutions. These traveling wave solutions are expressed in the form $u(x,t) = U(\xi)$, where $\xi = x - ct$.

Here, the space and time domains are represented by x and t, with the wave speed given by c. If $c = 0$, the resulting wave is called a standing wave. Such waves do not propagate and are typically observed when inducing a fixed boundary.

1.1.6 Types of Traveling Wave Solutions

The study of equations that model wave phenomena requires the study of displacement wave solutions. The traveling wave solution is a permanent-shape displacement solution with a constant speed. Traveling wave solutions are generally obtained by reducing nonlinear evolution equations to their associated ordinary differential equations. This is mainly handled using the ansatz $u(x,t) = U(\xi)$, where $\xi = x - ct$, and c is the wave speed, which transforms the PDE in x and t into an ordinary differential equation in ξ that can be solved by several appropriate methods. There are many types of traveling wave solutions that are of particular interest to soliton wave theory, which is rapidly developing in many scientific fields, from shallow water waves to plasma physics. As mentioned earlier, traveling waves appear in many types, and only some of these types will be addressed:

Solitary Waves and Solitons: Solitary waves are localized traveling waves moving at constant speeds and shapes, asymptotically vanishing at large distances. Solitons are special types of solitary waves. The soliton solution is a spatially localized solution, so $U'(\xi), U''(\xi)$, and $U'''(\xi) \to 0$ as $\xi \to \pm\infty$, $\xi = x - ct$. Solitons have the remarkable property of maintaining their identities when interacting with other solitons. The KdV equation is the pioneering model of the solitary wave, where the analytical solutions are bell-shaped. The soliton solution $u(x,t) = \operatorname{sech}^2(x - t)$, $-\pi \le x, t \le \pi$, has infinite support or infinite tails.

Periodic Solutions: Periodic solutions are traveling wave solutions that are periodic, such as $\cos(x - t)$. The standard wave equation $u_{tt} = u_{xx}$ yields periodic solutions. As noted earlier, because this standard wave equation is linear, it admits the d'Alembert solution, and the components can be superimposed. A periodic solution $u(x,t) = \cos(x - t)$, $-\pi \le x, t \le \pi$ for a standard wave equation.

-Kink Waves: Kink waves are traveling waves that rise or fall from one asymptotic state to another. The kink solution approaches a constant at infinity. The standard dissipative Burgers equation $u_t + uu_x = \mu u_{xx}$, where μ is the viscosity coefficient, is a well-known equation that yields kink solutions. Other equations also provide kink solutions, such as $u(x,t) = 1 - \tanh(x - t)$, $-10 \le x, t \le 10$ for the Burgers equation with $\mu = 0.5$.

Peakons: Peakons are peaked soliton wave solutions. In this case, the traveling wave solutions are smooth except for a peak at a corner of their crest. Peakons are the points at which the spatial derivative changes sign so that the peaks have a finite jump in the first derivative of the solution $u(x,t)$. This means that the peaks have discontinuities in the x-derivative, but the one-sided derivatives exist and differ only by a sign. Peakons are solitons that retain their shape and speed after interaction. Peakons have been studied and classified into periodic peakons and peakons with exponential decay. The integrable Camassa-Holm (CH) equation $u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xxt} + uu_{xxt}$ for $b = 2$ and $b = 3$, respectively, admits peaked soliton wave solutions. The (CH) equation has peaked soliton wave solutions of the form $u(x,t) = ce^{-|x-ct|}$.

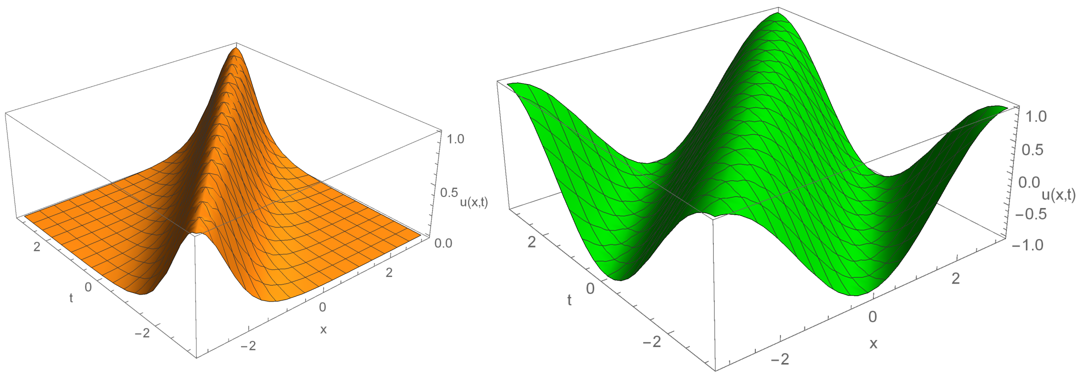
Cuspons: Cuspons are another form of solitons where the solution exhibits cusps at their crests. Unlike peakons, where the derivatives at the peak differ only by a sign, the derivatives at the jump of a cuspon diverge. Unfortunately, no explicit expression for cuspons could be found. Instead, a virtual expression was used to represent it graphically. The assumption is that a cuspon can be represented as $u(x,t) = ce^{-|x-ct|^k}$, $k > 1$.

Compactons: Compactons are a new class of solitons with compact spatial support such that each compacton is a soliton confined to a finite core. Compactons are defined by soliton waves with the remarkable property that after colliding with other compactons, they reappear with the same coherent shape. These particle-like waves exhibit elastic collisions similar to soliton collisions. It has been found that a compacton is a solitary wave with compact support where nonlinear dispersion confines it to a finite core, so the exponential wings vanish. The truly nonlinear dispersive $K(n,n)$ equations, a family of KdV-like nonlinear equations, are of the form $u_t + a(u^n)_x + (u^n)_{xx} = 0$, $a > 0$, $n > 1$, which support compact traveling soliton structures for $a > 0$. The definitions given so far for compactons are: 1. Compactons are solitons of finite wavelength. 2. Compactons are solitary waves with compact support. 3. Compactons are solitons without exponential tails. 4. Compactons are solitons characterized by the absence of infinite wings. 5. Compactons are robust soliton-like solutions.

An example of a compacton is $u(x,t) = \cos^2(x-t)$, $0 \leq x,t \leq 1$. The remarkable discovery of compactons has led to intense study in recent years. The study of compactons can provide insight into many scientific processes, such as super-deformed nuclei, cluster preformation in hydrodynamic models, fission of liquid droplets, and inertial fusion. Stability analysis has shown that compacton solutions are stable, where the stability condition is satisfied for arbitrary values of the nonlinearity parameter. The stability of compacton solutions has been studied using both linear stability and Lyapunov stability criteria. Moreover, compactons are non-analytic solutions, whereas classical solitons are analytic solutions. Solitons and compactons, with and without exponential wings, respectively, are named using the suffix ”-on” to indicate that they have particle-like properties, like phonons and photons.

Optical Soliton: An optical soliton is an electromagnetic pulse propagating without distortion. By its very nature, it is a stable solution to the propagation equation in the medium it traverses (typically an optical fiber). Theory shows that it has a hyperbolic secant shape. As an example, the Peregrine soliton is a mathematical solution to the Gross-Pitaevskii equation, which is equivalent to the Schrödinger equation with an added nonlinear term. This solution was established in 1983 by Howell Peregrine, a researcher in the mathematics department at the University of Bristol. It is used in optics, hydrodynamics, and plasma physics.

Example Curves for Different Types of Traveling Wave Solutions



(a) Graphic of soliton solution $\text{sech}^2(x - t), -\pi \leq x, t \leq \pi$ (b) Graphic of periodic solution $\cos(x - t), -\pi \leq x, t \leq \pi$

Figure 1.1: Surfaces of soliton solution and periodic solution , for $(x, t) \in [-\pi, \pi] \times [-\pi, \pi]$

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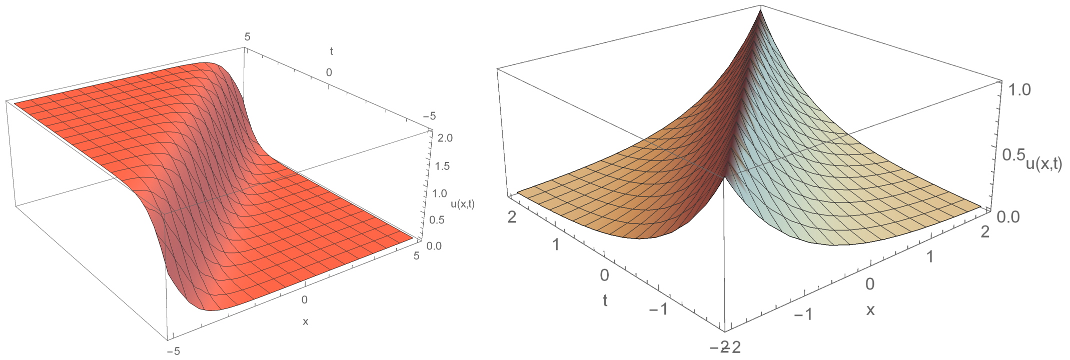

(a) Graphic of kink solution $1 - \tanh(x - t), -5 \leq x, t \leq 5$ (b) Graphic of peakon solution $\exp(-|x - t|), -2 \leq x, t \leq 2$

Figure 1.2: Surfaces of solution kink solution and peakon solution ,for $(x, t) \in [-5, 5]^2$ et $[-2, 2]^2$ respectively

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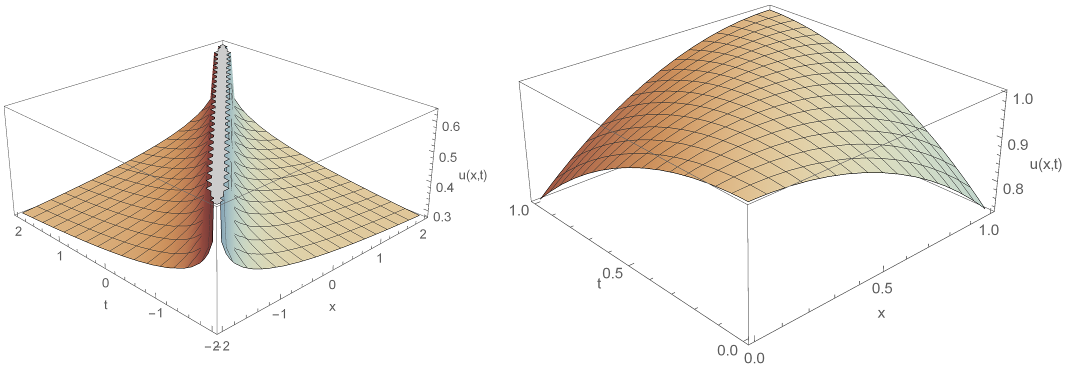

(a) Graphic of cuspon solution $\exp\left(-\sqrt[6]{|x-t|}\right), -2 \leq x, t \leq 2$

(b) Graphic of compacton solution $\cos^{\frac{1}{2}}(x-t), 0 \leq x, t \leq 1$

Figure 1.3: Surfaces of cuspon solution and solution compacton solution ,for $(x,t) \in [-2,2]^2$ et $[0,1]^2$ respectively

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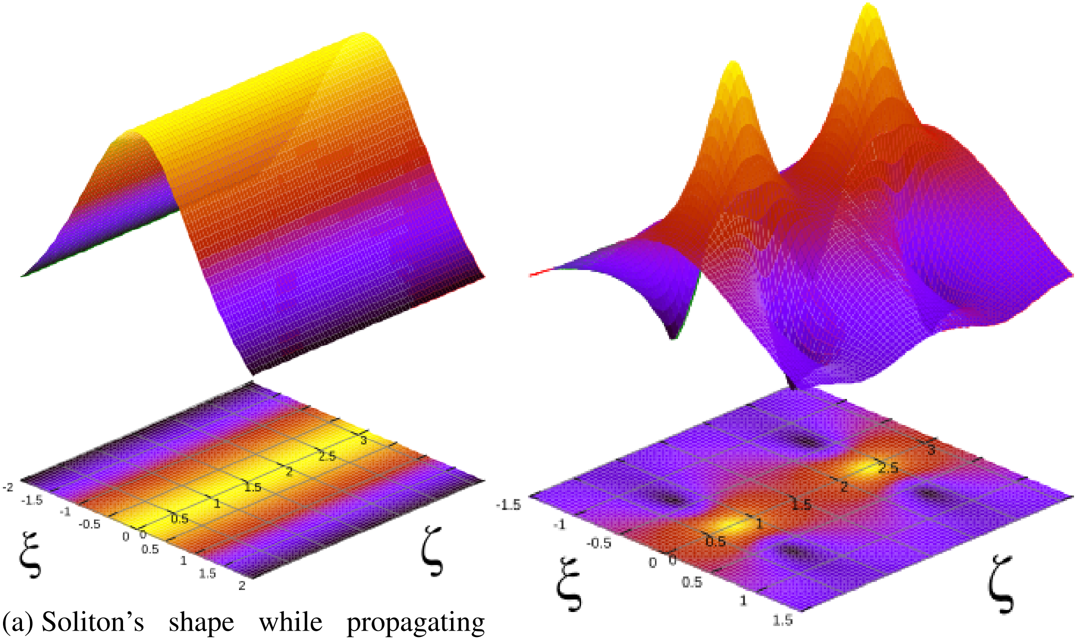


(a) Soliton's shape while propagating with $N = 1$, it does not change its shape

(b) Soliton's shape while propagating with $N = 2$, it changes its shape periodically

Figure 1.4: Surfaces of optical soliton solutions $\frac{1}{2}\frac{\partial^2 a}{\partial \xi^2} + i\frac{\partial a}{\partial \zeta} + N^2|a|^2 a = 0$.

1.2 Preliminary

1.2.1 An Overview of the generalized exponential rational function method

Let us state the main steps of GERFM [1, 3, 4, 5, 20, 21] as follows.

- **Step 1:** Let us take into account the nonlinear partial differential equation (NPDE) in the form given by

$$P(U, U_x, U_t, U_{xx}, \dots) = 0. \tag{1.2}$$

Using the transformation $U = U(\xi)$ and $\xi = kx - \omega t$, it is possible to reduce the NPDE to the following ordinary differential equation:

$$N(U, U', U'', \dots) = 0, \tag{1.3}$$

where the values of k and ω will be found later.

- **Step 2:** We look for a solution of Eq (1.3) with the following structure:

  $$U(\xi) = A_0 + \sum_{i=1}^{M} A_i \phi(\xi)^i, \tag{1.4}$$

  where

  $$\phi(\xi) = \frac{p_1 e^{q_1 \xi} + p_2 e^{q_2 \xi}}{p_3 e^{q_1 \xi} + p_4 e^{q_2 \xi}}. \tag{1.5}$$

  The values of constants p_i, q_i $(1 \leq i \leq 4)$, A_0, A_i and B_i $(1 \leq i \leq M)$ are constants to be determined. By considering the homogeneous balance principle, the value of M can be determined.

- **Step 3:** Substituting Eq (1.4) into Eq (1.3) and collecting all terms, the left-hand side of Eq (1.3) is converted to an algebraic equation $P(Z_1, Z_2, Z_3, Z_4) = 0$ in terms of $Z_j = e^{q_j \tau}$ for $(j = 1, ..., 4)$. Setting each coefficient of P to zero, a system of nonlinear equations is constructed.

- **Step 4:** solving the above algebraic equations using any symbolic computation software, the values of p_i, q_i $(1 \leq i \leq 4)$, A_0, A_i and B_i for $(1 \leq i \leq M)$ are determined.

  To get the solutions of the considered NPDE (1.2), we put the values of these parameters (Eq (1.4)).

  Group 1: We obtain $p = [i, -i, 1, 1]$ and $q = [i, -i, i, -i]$ so (1.5) turns to

  $$\phi(\xi) = -\frac{\sin \xi}{\cos \xi}.$$

  Group 2: We obtain $p = [-1, 0, 1, 1]$ and $q = [0, 0, 1, 0]$ so (1.5) turns to

$$\phi(\xi) = -\frac{1}{1+e^{\xi}}.$$

Group 3: We obtain $p = [i, -i, 1, 1]$ and $q = [i, -i, i, -i]$ so (1.5) turns to

$$\phi(\xi) = -\frac{\sin\xi}{\cos\xi}$$

Group 4: We obtain $p = [-1, 3, 1, -1]$ and $q = [1, -1, 1, -1]$so (1.5) turns to

$$\phi(\xi) = \frac{\cosh\tau - 2\,\sinh\tau}{\sinh\tau}.$$

Group 5: We obtain $p = [-3, -2, 1, 1]$ and $q = [0, 1, 0, 1]$so (1.5) turns to

$$\phi(\xi) = \frac{-3-2e^{\xi}}{1+e^{\xi}}.$$

Group 6: We obtain $p = [1, 2, 1, 1]$ and $q = [0, 1, 0, 1]$ so (1.5) turns to

$$\phi(\xi) = \frac{1+2e^{\xi}}{1+e^{\xi}}.$$

Group 7: We obtain $p = [-1-i, 1-i, -1, 1]$ and $q = [i, -i, i, -i]$ so (1.5) turns to

$$\phi(\xi) = \frac{\cos\xi + \sin\xi}{\sin\xi}.$$

Group 8: We obtain $p = [-2-i, 2-i, -1, 1]$ and $q = [i, -i, i, -i]$ so (1.5) turns to

$$\phi(\xi) = \frac{\cos\xi + 2\sin\xi}{\sin\xi}.$$

Group 9: We obtain $p = [1-i, -1-i, -1, 1]$ and $q = [i, -i, i, -i]$ so (1.5) turns to

$$\phi(\xi) = \frac{\cos\xi - \sin\xi}{\sin\xi}.$$

1.2.2 Description of the improved generalized Riccati equation mapping method

We suppose that a nonlinear PDE is in the following form:

$$P(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0 \tag{1.6}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its partial derivatives in which the highest order deivatives and nonlinear terms are involved. Let us now give te main stepes for solving Equation (1.6) using the improved Riccati equation method [6, 43, 41, 42, 44].

- **Step 1:** We look for its traveling wave solution in the form:

$$u(x,t) = U(\xi), \quad \xi = kx + \omega t \tag{1.7}$$

  where k, ω are constants. Substituting (1.7) into Equation (1.6) gives the nonlinear ODE for $U(\xi)$ as follows:

$$H(u, u', u'', ...) = 0 \tag{1.8}$$

  where H is a polynomial in $U(\xi)$ and its total derivatives $u', u'', u''', \ldots$, such that $U' = \frac{dU}{d\xi}$ and $U'' = \frac{d^2U}{d\xi^2} \ldots$.

- **Step 2:**We suppose that the solution of the ODE (1.8) can be expressed as follows:

$$U(\xi) = \sum_{i=-m}^{m} a_i Q^i(\xi) \tag{1.9}$$

  where a_i $(i = 0, \pm1, \pm2, \ldots, \pm m)$ are constants to be determined such as $a_m \neq 0$, and $Q = Q(\xi)$ is the solution of generalized Riccati equation:

$$Q' = r + pQ + qQ^2 \tag{1.10}$$

  where r, p, q are constants, such that $q \neq 0$.

- **Step 3**: We determine the positive integer m in (1.9) by balancing the nonlinear terms and the highest order derivatives of $u(\xi)$ in Equation (1.8).

- **Step 4:** substituting (1.9) and along with Equation (1.10) into Equation (1.8) and then equation all the coefficients of Q^i $(i = 0, \pm1, \pm2, \ldots, \pm m)$ to zero yield a system of algebraic equations which can be solved by using the Maple or Mathematica to find the values of the onstants $a_i(-m, \ldots, m)$ and k, ω.

- **Step 5:** It is well-known that Equation (1.10) has many families of solutions as follows:
- **Type 1:** When $\Delta = p^2 - 4qr > 0$ and $pq \neq 0$ or $qr \neq 0$ We have:

$$
\begin{aligned}
\Phi_1(\xi) &= -\frac{1}{2q}\left[p + \sqrt{\Lambda}\tanh\left(\frac{\sqrt{\Lambda}}{2}\xi\right)\right],\\
\Phi_2(\xi) &= -\frac{1}{2q}\left[p + \sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right],\\
\Phi_3(\xi) &= -\frac{1}{2q}\left[p + \sqrt{\Delta}\left(\tanh(\sqrt{\Delta}\xi) \pm i\,\mathrm{sech}(\sqrt{\Delta}\xi)\right)\right], \quad i = \sqrt{-1},\\
\Phi_4(\xi) &= -\frac{1}{2q}\left[p + \sqrt{\Delta}\left(\coth(\sqrt{\Delta}\xi) \pm \mathrm{csch}(\sqrt{\Delta}\xi)\right)\right],\\
\Phi_5(\xi) &= -\frac{1}{4q}\left[2p + \sqrt{\Delta}\left(\tanh\left(\frac{\sqrt{\Delta}}{4}\xi\right) \pm \coth\left(\frac{\sqrt{\Delta}}{4}\xi\right)\right)\right],\\
\Phi_6(\xi) &= \frac{1}{2q}\left[-p + \frac{\sqrt{\Delta(A^2+B^2)} - A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A\sinh(\sqrt{\Delta}\xi) + B}\right],\\
\Phi_7(\xi) &= \frac{1}{2q}\left[-p - \frac{\sqrt{\Delta(B^2-A^2)} + A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A\sinh(\sqrt{\Delta}\xi) + B}\right].
\end{aligned}
$$

where A and B are two non-zero real constants satisfying $B^2 - A^2 > 0$.

$$
\begin{aligned}
\Phi_8(\xi) &= \frac{2r\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - p\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)},\\
\Phi_9(\xi) &= \frac{-2r\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{p\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - \sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)},\\
\Phi_{10}(\xi) &= \frac{2r\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right) - p\cosh\left(\sqrt{\Delta}\xi\right) \pm i\sqrt{\Delta}}, \quad i = \sqrt{-1},\\
\Phi_{11}(\xi) &= \frac{2r\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{-p\sinh\left(\sqrt{\Delta}\xi\right) + \sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right) \pm \sqrt{\Delta}},\\
\Phi_{12}(\xi) &= \frac{4r\sinh\left(\frac{\sqrt{\Delta}}{4}\xi\right)\cosh\left(\frac{\sqrt{\Delta}}{4}\xi\right)}{-2p\sinh\left(\frac{\sqrt{\Delta}}{4}\xi\right)\cosh\left(\frac{\sqrt{\Delta}}{4}\xi\right) + 2\sqrt{\Delta}\cosh^2\left(\frac{\sqrt{\Delta}}{2}\xi\right) - \sqrt{\Delta}}.
\end{aligned}
$$

Type 2: When $\Delta = p^2 - 4qr < 0$ and $pq \neq 0$ or $qr \neq 0$ We have:

$$
\begin{aligned}
\Phi_{13}(\xi) &= \frac{1}{2q}\left[-p + \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right],\\
\Phi_{14}(\xi) &= -\frac{1}{2q}\left[p + \sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right],\\
\Phi_{15}(\xi) &= \frac{1}{2q}\left[-p + \sqrt{-\Delta}\left(\tan(\sqrt{-\Delta}\xi) \pm \sec(\sqrt{-\Delta}\xi)\right)\right],\\
\Phi_{16}(\xi) &= -\frac{1}{2q}\left[p + \sqrt{-\Delta}\left(\cot(\sqrt{-\Delta}\xi) \pm \csc(\sqrt{-\Delta}\xi)\right)\right],\\
\Phi_{17}(\xi) &= \frac{1}{4q}\left[-2p + \sqrt{-\Delta}\left(\tan\left(\frac{\sqrt{-\Delta}}{4}\xi\right) - \cot\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\right)\right],\\
\Phi_{18}(\xi) &= \frac{1}{2q}\left[-p + \frac{\pm\sqrt{-\Delta}(A^2 - B^2) - A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi) + B}\right],\\
\Phi_{19}(\xi) &= \frac{1}{2q}\left[-p - \frac{\pm\sqrt{-\Delta}(A^2 - B^2) - A\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi) + B}\right].
\end{aligned}
$$

where A and B are two non-zero real constants satisfying $A^2 + B^2 > 0$.

$$
\begin{aligned}
\Phi_{20}(\xi) &= -\frac{2r\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + p\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)},\\
\Phi_{21}(\xi) &= \frac{2r\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{-p\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + \sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)},\\
\Phi_{22}(\xi) &= -\frac{2r\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right) + p\cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}},\\
\Phi_{23}(\xi) &= \frac{2r\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{-p\sin\left(\sqrt{-\Delta}\xi\right) + \sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}},\\
\Phi_{24}(\xi) &= \frac{4r\sin\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\cos\left(\frac{\sqrt{-\Delta}}{4}\xi\right)}{-2p\sin\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\cos\left(\frac{\sqrt{-\Delta}}{4}\xi\right) + 2\sqrt{-\Delta}\cos^2\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - \sqrt{-\Delta}}.
\end{aligned}
$$

- **Type 3:** When $r = 0$ and $pq \neq 0$ we have:

$$\begin{aligned}\Phi_{25}(\xi) &= \frac{-p\,d}{q\left[d+\cosh(p\xi)-\sinh(p\xi)\right]},\\ \Phi_{26}(\xi) &= -\frac{p\left[\cosh(p\xi)+\sinh(p\xi)\right]}{q\left[d+\cosh(p\xi)+\sinh(p\xi)\right]},\end{aligned}$$

where d is an arbitrary constant

- **Type 4:** When $r = p = 0$ and $q \neq 0$

$$\Phi Q_{27}(\xi) = \frac{-1}{q\xi + c_1}$$

where c is an arbitrary constant.

- **Step 6:** Substituting the well know solutions of Equation (1.10) listed above in step 5 into (1.9) we have many families of exact solution of Equation (1.6)

1.2.3 Description of the Modified Exponential Function Method

[33, 34, 35, 36, 37, 38, 39]
The general nonlinear evolution equation, say in $(1 + 1)$ independent variables x and t, is given by:

$$P(u, D_t u, u_x, D_t^2 u, u_{xx}, D_t u_x, \ldots) = 0, \tag{1.11}$$

where $u = u(x,t)$ is an unknown function, and P is a polynomial in u and its partial derivatives, including nonlinear terms and the highest-order derivatives. To find the exact solution of Eq (1.11) using the Exp($-\Phi(\eta)$)-expansion method, we follow these steps:

- **Step 1:** To obtain the exact solution, the following complex transformation is applied:

$$u(x,t) = U(\eta), \quad \eta = kx - \omega t, \tag{1.12}$$

where k and ω are constants to be determined later. Then, Eq (1.11) reduces to the following nonlinear ordinary differential equation (ODE):

$$N(U, -\omega U', kU', \omega^2 U'', k^2 U'', -\omega k U', \ldots) = 0, \tag{1.13}$$

where $U^{(n)} = \frac{d^n U}{d\eta^n}$.

- **Step 2:** Assume that the solution of Eq (1.13) can be expressed as a finite power series of the form:

$$U(\eta) = \frac{\sum_{i=0}^{N} a_i(\exp(-\Phi(\eta))^i}{\sum_{j=0}^{M} b_j(\exp(-\Phi(\eta))^j}, \tag{1.14}$$

where $a_0, a_1, \ldots, a_N$ ($a_N \neq 0$) and $b_0, b_1, \ldots, b_M$ ($b_M \neq 0$) are constants to be determined later. Let $\Phi = \Phi(\eta)$ satisfy the differential equation:

$$\Phi'(\eta) = \mu e^{\Phi(\eta)} + e^{-\Phi(\eta)} + \lambda, \tag{1.15}$$

where μ and λ are constants to be discussed later.

The general solutions of (1.15) can be written in the following forms:

$$\Phi(\eta) = \begin{cases} \log\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\eta+C)\right)-\lambda}{2\mu}\right), & \lambda^2-4\mu>0, \mu\neq 0, \\ \log\left(\frac{\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\eta+C)\right)-\lambda}{2\mu}\right), & \lambda^2-4\mu<0, \mu\neq 0, \\ \log\left(\frac{-2(\lambda(\eta+C)+2)}{\lambda^2(\eta+C)}\right), & \lambda^2-4\mu=0, \mu\neq 0, \lambda\neq 0, \\ -\log\left(\frac{\lambda}{\sinh(\lambda(\eta+C))+\cosh(\lambda(\eta+C))-1}\right), & \lambda^2-4\mu\neq 0, \mu=0, \lambda\neq 0, \\ \log(\eta+C), & \lambda^2-4\mu=0, \mu=0, \lambda=0, \end{cases} \tag{1.16}$$

where C is an arbitrary real constant.

- **Step 3:** The positive integers N and M can be determined by considering the homogeneous balance between the highest-order nonlinear term and the highest-order derivative term in Eq (1.12). Suppose the degree of $U(\eta)$ is $N-M$, then the degree of other expressions can be evaluated as follows:

$$D(U^{(p)}) = N - M + p, \quad D(U^p(U^{(q)})^s) = (N-M)p + s(N-M+q).$$

- **Step 4:** Substitute Eq (1.14) using Eq (1.15) into Eq (1.13). Then, collect the coefficients of the same power of $e^{n\Phi(\eta)}$ ($n = 0, 1, 2, \ldots$). A system of nonlinear algebraic equations is obtained by setting each coefficient to zero. The resulting algebraic system is solved using **Mathematica** to determine the values of the unknown constants $a_0, a_1, \dots, a_N$, $b_0, b_1, \dots, b_M$, k, and ω.

- **Step 5:** Substitute a_n, k, and ω into (1.14) using (1.16) to obtain all exact solutions of the nonlinear evolution equation (1.11).

1.2.4 Description of the Extended $\left(\frac{G'}{G}\right)$-Expansion Method

[43, 32, 40]

The general nonlinear evolution equation, say in two independent variables x and t, is given by:

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, u_{xxx}, \dots) = 0, \tag{1.17}$$

where $u = u(x,t)$ is an unknown function, and P is a polynomial in u and its partial derivatives, including nonlinear terms and the highest-order derivatives. To find the traveling wave solution of Eq (1.17) using the extended $\left(\frac{G'}{G}\right)$-expansion method, we follow these steps:

- **Step 1:** To obtain an exact traveling wave solution, the following complex transformation is applied:

  $$u(x,t) = U(\xi), \quad \xi = kx - \omega t, \tag{1.18}$$

  where k and ω are constants to be determined later. Then, Eq (1.17) reduces to the following nonlinear ODE:

  $$P(U, -\omega U', kU', \omega^2 U'', k^2 U'', -\omega k U', \dots) = 0, \tag{1.19}$$

  where $U^{(n)} = \frac{d^n U}{d\xi^n}$.

- **Step 2:** Assume that the solution of Eq (1.19) can be expressed as a finite power series of the form:

  $$U(\xi) = \sum_{n=-N}^{N} a_n \left(\frac{G'}{G}\right)^n, \tag{1.20}$$

  where $G = G(\xi)$ satisfies the second-order linear ODE:

  $$G'' + \lambda G' + \mu G = 0, \tag{1.21}$$

where λ and μ are constants to be discussed later, and $a_0, a_1, \ldots, a_N$ $(a_N \neq 0)$ are constants to be determined later.

The general solutions of (1.21) can be written in the following forms:

$$G(\xi) = \begin{cases} e^{-\frac{1}{2}\lambda\xi}\left(A_2 \sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right) + A_1 \cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)\right), & \lambda^2-4\mu > 0, \\ e^{-\frac{1}{2}\lambda\xi}\left(A_2 \sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right) + A_1 \cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)\right), & \lambda^2-4\mu < 0, \\ (A_2\xi + A_1)e^{-\frac{1}{2}\lambda\xi}, & \lambda^2-4\mu = 0, \end{cases} \tag{1.22}$$

which gives:

$$\frac{G'}{G} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{A_1\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}{A_1\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}\right) - \frac{\lambda}{2}, & \lambda^2-4\mu > 0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{A_2\cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)+A_1\sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}{A_2\sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)-A_1\cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}\right) - \frac{\lambda}{2}, & \lambda^2-4\mu < 0, \\ \frac{A_2}{A_2\xi+A_1} - \frac{\lambda}{2}, & \lambda^2-4\mu = 0, \end{cases} \tag{1.23}$$

where A_1 and A_2 are arbitrary constants.

- **Step 3:** The degree N of the power series (1.20) is determined by considering the homogeneous balance between the highest-order nonlinear term and the highest-order derivative term in Eq (1.19).

- **Step 4:** Substitute Eq (1.20) using Eq (1.21) into Eq (1.19). Then, collect the coefficients of the same power of $\left(\frac{G'}{G}\right)^n$ $(n = 0, \pm1, \pm2, \ldots, N)$. A system of nonlinear algebraic equations is obtained by setting each coefficient to zero. The resulting algebraic system is solved using **Mathematica** to determine the values of the unknown constants $a_0, a_1, \ldots, a_N$, k, and ω.

- **Step 5:** Since the general solution of (1.21) is now known, substitute a_n, k, ω, and (1.23) into (1.20) to obtain three types of exact traveling wave solutions of the nonlinear evolution equation (1.17).

Chapter 2

Application of generalized exponential rational function method to the improved Boussinesq equation:

The improved Boussinesq equation [10, 11, 12, 13, 14] is a refined version of the classical Boussinesq equation, which models nonlinear wave propagation in dispersive media, particularly in shallow water waves and elastic solids. Its general form is of the shape

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2} - u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} - \left(\frac{\partial u(x,t)}{\partial x}\right)^2 = 0. \tag{2.1}$$

We will Now attempt to find exact solution using the **GERFM**

$$k^2\left(-\omega^2\right)U''(\xi) - \frac{1}{2}k^2U^2(\xi) + \left(\omega^2 - k^2\right)U(\xi) = 0 \tag{2.2}$$

Balancing U^2 with U'', we get $N = 2$

So the solution is in the form

$$U(\xi) = a_2\phi(\xi)^2 + a_1\phi(\xi) + a_0 \tag{2.3}$$

Where

$$\phi(\xi) = \frac{p_1e^{q_1\xi} + p_2e^{q_2\xi}}{p_3e^{q_1\xi} + p_4e^{q_2\xi}}. \tag{2.4}$$

To get the solutions of the considered NPDE (1.2), we put the values of these parameters (Eq (1.4)).

Group 1: We obtain $p = [i, -i, 1, 1]$ and $q = [i, -i, i, -i]$ so (2.4) turns to

$$\phi(\xi) = -\tan(\xi). \tag{2.5}$$

By using the **Mathematica** software, we obtain a polynomial. By setting the coefficients of this polynomial corresponding to the same powers to zero, we derive a system of algebraic equations for a_0, a_1, a_2, ω .

$$\begin{aligned}
&-2a_2k^2\omega^2 - \frac{1}{2}a_0^2k^2 - a_0k^2 + a_0\omega^2 = 0,\\
&2a_1k^2\omega^2 + a_0a_1k^2 + a_1k^2 - a_1\omega^2 = 0,\\
&-8a_2k^2\omega^2 - \frac{1}{2}a_1^2k^2 - a_0a_2k^2 - a_2k^2 + a_2\omega^2 = 0,\\
&2a_1k^2\omega^2 + a_1a_2k^2 = 0,\\
&-6a_2k^2\omega^2 - \frac{1}{2}a_2^2k^2 = 0.
\end{aligned} \tag{2.6}$$

After solving the resulting algebraic system and with help of **Mathematica**, we obtain the following results:

Case 1:

$$\left\{a_0 \to \frac{4k^2}{4k^2-1}, a_1 \to 0, a_2 \to \frac{4k^2}{4k^2-1}, \omega \to \pm\frac{k}{\sqrt{1-4k^2}}\right\} \tag{2.7}$$

Putting these results in Eq (2.3) and (2.5) leads to

$$U(\xi) = \frac{4k^2\sec^2(\xi)}{4k^2-1} \tag{2.8}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{4k^2\sec^2\left(k\left(\frac{t}{\sqrt{1-4k^2}} \mp x\right)\right)}{4k^2-1} \tag{2.9}$$

where $\xi = kx - \omega t$

Case 2:

$$\left\{a_0 \to -\frac{12k^2}{4k^2+1}, a_1 \to 0, a_2 \to -\frac{12k^2}{4k^2+1}, \omega \to \pm\frac{k}{\sqrt{4k^2+1}}\right\} \tag{2.10}$$

Putting these results in Eq (2.3) and (2.5) leads to

$$U(\xi) = -\frac{12k^2\sec^2(\xi)}{4k^2+1} \tag{2.11}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = -\frac{12k^2 \sec^2\left(k\left(\frac{1}{\sqrt{4k^2+1}} \mp x\right)\right)}{4k^2+1} \tag{2.12}$$

where $\xi = kx - \omega t$

Case 3:

$$\left\{a_0 \to -\frac{3}{2}, a_1 \to 0, a_2 \to -\frac{3}{2}, k \to -\frac{1}{2}, \omega \to \pm\frac{1}{2\sqrt{2}}\right\} \tag{2.13}$$

Putting these results in Eq (2.3) and (2.5) leads to

$$U(\xi) = \frac{1}{2}(-3)\sec^2(\xi) \tag{2.14}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = -\frac{3}{\cos\left(\frac{t}{\sqrt{2}} \pm x\right) + 1} \tag{2.15}$$

where $\xi = kx - \omega t$

Case 4:

$$\left\{a_0 \to -\frac{3}{2}, a_1 \to 0, a_2 \to -\frac{3}{2}, k \to \frac{1}{2}, \omega \to \pm\frac{1}{2\sqrt{2}}\right\} \tag{2.16}$$

Putting these results in Eq (2.3) and (2.5) leads to

$$U(\xi) = \frac{1}{2}(-3)\sec^2(\xi) \tag{2.17}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = -\frac{3}{\cos\left(\frac{t}{\sqrt{2}} \mp x\right) + 1} \tag{2.18}$$

where $\xi = kx - \omega t$

Case 5:

$$\left\{a_0 \to \frac{1}{2}, a_1 \to 0, a_2 \to \frac{3}{2}, k \to -\frac{i}{2}, \omega \to \pm\frac{i}{2\sqrt{2}}\right\} \tag{2.19}$$

Putting these results in Eq (2.3) and (2.5) leads to

$$U(\xi) = \frac{3\sec^2(\xi)}{2} - 1 \tag{2.20}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{3}{\cosh\left(\frac{t}{\sqrt{2}} \pm x\right) + 1} - 1 \tag{2.21}$$

where $\xi = kx - \omega t$

Case 6:

$$\left\{a_0 \to \frac{1}{2}, a_1 \to 0, a_2 \to \frac{3}{2}, k \to \frac{i}{2}, \omega \to \pm\frac{i}{2\sqrt{2}}\right\} \tag{2.22}$$

Putting these results in Eq (2.3) and (2.5) leads to

$$U(\xi) = \frac{3\sec^2(\xi)}{2} - 1 \tag{2.23}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{3}{\cosh\left(\frac{t}{\sqrt{2}} \mp x\right) + 1} - 1 \tag{2.24}$$

where $\xi = kx - \omega t$

Group 2: We obtain $p = [-1, 0, 1, 1]$ and $q = [0, 0, 1, 0]$ so (2.4) turns to

$$\phi(\xi) = -\frac{1}{e^{\xi} + 1}. \tag{2.25}$$

By using the **Mathematica** software, we obtain a polynomial and By setting the coefficients of this polynomial corresponding to the same powers to zero, we derive a system of algebraic equations for a_0, a_1, a_2, ω .

$$\begin{aligned}
&a_0^2k^2 + a_1^2k^2 + a_2^2k^2 + 2a_0k^2 - 2a_0a_1k^2 - 2a_1k^2 + 2a_0a_2k^2 - 2a_1a_2k^2 + 2a_2k^2 - 2a_0\omega^2 + 2a_1\omega^2 - 2a_2\omega^2 = 0,\\
&a_0^2k^2 + 2a_0k^2 - 2a_0\omega^2 = 0,\\
&-2a_1k^2\omega^2 + 4a_0^2k^2 + 8a_0k^2 - 2a_0a_1k^2 - 2a_1k^2 - 8a_0\omega^2 + 2a_1\omega^2 = 0,\\
&8a_2k^2\omega^2 + 6a_0^2k^2 + a_1^2k^2 + 12a_0k^2 - 6a_0a_1k^2 - 6a_1k^2 + 2a_0a_2k^2 + 2a_2k^2 - 12a_0\omega^2 + 6a_1\omega^2 - 2a_2\omega^2 = 0,\\
&2a_1k^2\omega^2 - 4a_2k^2\omega^2 + 4a_0^2k^2 + 2a_1^2k^2 + 8a_0k^2 - 6a_0a_1k^2 - 6a_1k^2\\
&+4a_0a_2k^2 - 2a_1a_2k^2 + 4a_2k^2 - 8a_0\omega^2 + 6a_1\omega^2 - 4a_2\omega^2 = 0.
\end{aligned} \tag{2.26}$$

After solving the resulting algebraic system and with help of **Mathematica**, we obtain the following results:

Case 1:

$$\left\{a_0 \to 0, a_1 \to \frac{12k^2}{k^2-1}, a_2 \to \frac{12k^2}{k^2-1}, \omega \to \pm\frac{k}{\sqrt{1-k^2}}\right\} \tag{2.27}$$

Putting these results in Eq (2.3) and (2.25) leads to

$$U(\xi) = -\frac{6k^2}{(k^2-1)(\cosh(\xi)+1)} \tag{2.28}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u_1(x,t) = -\frac{3k^2\operatorname{sech}^2\left(\frac{1}{2}k\left(\frac{t}{\sqrt{k^2+1}} \mp x\right)\right)}{k^2-1} \tag{2.29}$$

where $\xi = kx - \omega t$

Case 2:

$$\left\{a_0 \to -\frac{2k^2}{k^2+1}, a_1 \to -\frac{12k^2}{k^2+1}, a_2 \to -\frac{12k^2}{k^2+1}, \omega \to \pm\frac{k}{\sqrt{k^2+1}}\right\} \tag{2.30}$$

Putting these results in Eq (2.3) and (2.25) leads to

$$U(\xi) = -\frac{2k^2(\cosh(\xi)-2)}{(k^2+1)(\cosh(\xi)+1)} \tag{2.31}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{k^2\left(3\operatorname{sech}^2\left(\frac{1}{2}k\left(\frac{t}{\sqrt{k^2+1}} \mp x\right)\right) - 2\right)}{k^2+1} \tag{2.32}$$

where $\xi = kx - \omega t$

Group 3: We obtain $p = [i, -i, 1, 1]$ and $q = [i, -i, i, -i]$ so (2.4) turns to

$$\phi(\xi) = -\tan(\xi) \tag{2.33}$$

In this group, we find the solutions as those in Group 1.

Group 4: We obtain $p = [-1, 3, 1, -1]$ and $q = [1, -1, 1, -1]$so (2.4) turns to

$$\phi(\xi) = \frac{1}{2}(\coth(\xi) - 2). \tag{2.34}$$

By using the **Mathematica** software, we obtain a polynomial and By setting the coefficients of this polynomial corresponding to the same powers to zero, we derive a system of algebraic equations for a_0, a_1, a_2, ω .

$$\begin{aligned}
&-16a_2k^2(\omega^2+2) - 32a_0(k^2-\omega^2) - 16a_0^2k^2 - 16a_1^2k^2 - 16a_2^2k^2 + 32a_0a_1k^2 = 0,\\
&32a_1k^2 - 32a_0a_2k^2 + 32a_1a_2k^2 - 32a_1\omega^2 + 32a_2\omega^2 = 0,\\
&-8a_2\left(k^2(1-8\omega^2) - \omega^2\right) - 4a_1^2k^2 - 24a_2^2k^2 - 8a_0a_2k^2 + 24a_1a_2k^2 = 0,\\
&-48a_2k^2\omega^2 - a_2^2k^2 = 0,\\
&-32a_1k^2\omega^2 + 64a_2k^2\omega^2 + 8a_2^2k^2 - 4a_1a_2k^2 = 0,\\
&16a_1(k^2(2\omega^2-1)+\omega^2) - 32a_2(k^2(2\omega^2-1)+\omega^2) = 0,\\
&+16a_1^2k^2 + 32a_2^2k^2 - 16a_0a_1k^2 + 32a_0a_2k^2 - 48a_1a_2k^2 = 0.
\end{aligned} \tag{2.35}$$

After solving the resulting algebraic system and with help of **Mathematica**, we obtain the followin results:

Case 1:

$$\left\{a_0 \to \frac{36k^2}{4k^2-1}, a_1 \to \frac{96k^2}{4k^2-1}, a_2 \to \frac{48k^2}{4k^2-1}, \omega \to \pm\frac{k}{\sqrt{1-4k^2}}\right\} \tag{2.36}$$

Putting these results in Eq (2.3) and (2.37) leads to

$$U(\xi) = \frac{12k^2\operatorname{csch}^2(\xi)}{4k^2-1} \tag{2.37}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{12k^2\operatorname{csch}^2\left(k\left(\frac{t}{\sqrt{1-4k^2}} \mp x\right)\right)}{4k^2-1} \tag{2.38}$$

where $\xi = kx - \omega t$

Case 2:

$$\left\{a_0 \to -\frac{44k^2}{4k^2+1}, a_1 \to -\frac{96k^2}{4k^2+1}, a_2 \to -\frac{48k^2}{4k^2+1}, \omega \to \pm\frac{k}{\sqrt{4k^2+1}}\right\} \tag{2.39}$$

Putting these results in Eq (2.3) and (2.37) leads to

$$U(\xi) = \frac{4k^2\left(1 - 3\coth^2\left(k\left(\frac{t}{\sqrt{4k^2+1}} \mp x\right)\right)\right)}{4k^2+1} \tag{2.40}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{4k^2\left(1 - 3\coth^2\left(kx \mp \frac{kt}{\sqrt{4k^2+1}}\right)\right)}{4k^2+1} \tag{2.41}$$

where $\xi = kx - \omega t$

Case 3:

$$\left\{a_0 \to -\frac{11}{2}, a_1 \to -12, a_2 \to -6, \omega \to \pm\frac{1}{2\sqrt{2}}, k \to -\frac{1}{2}\right\} \tag{2.42}$$

Putting these results in Eq (2.3) and (2.37) leads to

$$U(\xi) = -\frac{2k^2(\cosh(\xi) - 2)}{(k^2+1)(\cosh(\xi)+1)} \tag{2.43}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = -\frac{3}{\cosh\left(\frac{t}{\sqrt{2}} \pm x\right) - 1} - 1 \tag{2.44}$$

where $\xi = kx - \omega t$

Case 4:

$$\left\{a_0 \to \frac{9}{2}, a_1 \to 12, a_2 \to 6, \omega \to \pm\frac{i}{2\sqrt{2}}, k \to -\frac{i}{2}\right\} \tag{2.45}$$

Putting these results in Eq (2.3) and (2.37) leads to

$$U(\xi) = \frac{3\operatorname{csch}^2(\xi)}{2} \tag{2.46}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{3}{\cos\left(\frac{t}{\sqrt{2}} \pm x\right) - 1} \tag{2.47}$$

where $\xi = kx - \omega t$

Case 5:

$$\left\{a_0 \to \frac{9}{2}, a_1 \to 12, a_2 \to 6, \omega \to \pm\frac{i}{2\sqrt{2}}, k \to \frac{i}{2}\right\} \tag{2.48}$$

Putting these results in Eq (2.3) and (2.37) leads to

$$U(\xi) = \frac{3\operatorname{csch}^2(\xi)}{2} \tag{2.49}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{3}{\cos\left(\frac{t}{\sqrt{2}} \mp x\right) - 1} \tag{2.50}$$

where $\xi = kx - \omega t$

Case 6:

$$\left\{a_0 \to -\frac{11}{2}, a_1 \to -12, a_2 \to -6, \omega \to \pm\frac{1}{2\sqrt{2}}, k \to \frac{1}{2}\right\} \tag{2.51}$$

Putting these results in Eq (2.3) and (2.37) leads to

$$U(\xi) = -\frac{1}{2}3\operatorname{csch}^2(\xi) - 1 \tag{2.52}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = -\frac{3}{\cosh\left(\frac{t}{\sqrt{2}} \mp x\right) - 1} - 1 \tag{2.53}$$

where $\xi = kx - \omega t$

Group 5: We obtain $p = [-3, -2, 1, 1]$ and $q = [0, 1, 0, 1]$so (2.4) turns to

$$\phi(\xi) = \frac{-3 - 2e^{\tau}}{1 + e^{\tau}}. \tag{2.54}$$

By using the **Mathematica** software, we obtain a polynomial and By setting the coefficients of this polynomial corresponding to the same powers to zero, we derive a system of algebraic equations for a_0, a_1, a_2, ω .

$$\begin{aligned}
&a_0^2k^2+9a_1^2k^2+81a_2^2k^2+a_0k^2-6a_0a_1k^2-6a_1k^2+18a_0a_2k^2-54a_1a_2k^2+18a_2k^2-2a_0\omega^2\\
&+6a_1\omega^2-18a_2\omega^2=0,\\
&a_0^2k^2+4a_1^2k^2+16a_2^2k^2+2a_0k^2-4a_0a_1k^2-4a_1k^2+8a_0a_2k^2-16a_1a_2k^2+8a_2k^2-2a_0\omega^2+\\
&4a_1\omega^2-8a_2\omega^2=0,\\
&-2a_1k^2\omega^2+8a_2k^2\omega^2+4a_0^2k^2+20a_1^2k^2+96a_2^2k^2+8a_0k^2-18a_0a_1k^2-18a_1k^2+40a_0a_2k^2\\
&-88a_1a_2k^2+40a_2k^2-8a_0\omega^2+18a_1\omega^2-40a_2\omega^2=0,\\
&8a_2k^2\omega^2+6a_0^2k^2+37a_1^2k^2+216a_2^2k^2+12a_0k^2-30a_0a_1k^2-30a_1k^2+74a_0a_2k^2-180a_1a_2k^2\\
&+74a_2k^2-12a_0\omega^2+30a_1\omega^2-74a_2\omega^2=0,\\
&2a_1k^2\omega^2-12a_2k^2\omega^2+4a_0^2k^2+30a_1^2k^2+216a_2^2k^2+8a_0k^2-22a_0a_1k^2-22a_1k^2+60a_0a_2k^2\\
&-162a_1a_2k^2+60a_2k^2-8a_0\omega^2+22a_1\omega^2-60a_2\omega^2=0
\end{aligned} \tag{2.55}$$

After solving the resulting algebraic system and with help of **Mathematica**, we obtain the followin results:

Case 1:

$$\left\{\to \frac{72k^2}{k^2-1}, a_1 \to -\frac{60k^2}{k^2-1}, a_2 \to \frac{12k^2}{k^2-1}, \omega \to \pm\frac{k}{\sqrt{1-k^2}}\right\} \tag{2.56}$$

Putting these results in Eq (2.3) and (2.54) leads to

$$U(\xi) = \frac{12k^2\left(4e^{\xi}+5\right)\left(5e^{\xi}+6\right)}{(k^2-1)\,(e^{\xi}+1)^2} \tag{2.57}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{3k^2\left(\tanh\left(\frac{1}{2}k\left(\frac{t}{\sqrt{1-k^2}} \mp x\right)\right)+\pm 9\right)\left(\tanh\left(\frac{1}{2}k\left(\frac{t}{\sqrt{1-k^2}} \mp x\right)\right)+\pm 11\right)}{k^2-1} \tag{2.58}$$

where $\xi = kx - \omega t$

Case 2:

$$\left\{a_0 \to -\frac{74k^2}{k^2+1}, a_1 \to \frac{60k^2}{k^2+1}, a_2 \to -\frac{12k^2}{k^2+1}, \omega \to \pm\frac{k}{\sqrt{k^2+1}}\right\} \tag{2.59}$$

Putting these results in Eq (2.3) and (2.54) leads to

$$U(\xi) = \frac{k^2 \operatorname{sech}^2\left(\frac{\xi}{2}\right)(30\sinh(\xi) - 151\cosh(\xi) - 148)}{k^2+1} \tag{2.60}$$

When $\omega \to \frac{k}{\sqrt{k^2+1}}$ Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u_1(x,t) = \frac{k^2\left(\frac{60\sinh\left(k\left(x-\frac{t}{\sqrt{1-k^2}}\right)\right)+6}{\cosh\left(k\left(x-\frac{t}{\sqrt{1-k^2}}\right)\right)+1} - 302\right)}{k^2+1} \tag{2.61}$$

where $\xi = kx - \omega t$

And when $\omega \to -\frac{k}{\sqrt{k^2+1}}$ Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u_2(x,t) = \frac{k^2\left(60\tanh\left(\frac{1}{2}k\left(\frac{t}{\sqrt{1-k^2}}+x\right)\right) + 3\operatorname{sech}^2\left(\frac{1}{2}k\left(\frac{t}{\sqrt{1-k^2}}+x\right)\right) - 302\right)}{k^2+1} \tag{2.62}$$

Group 6: We obtain $p = [1,2,1,1]$ and $q = [0,1,0,1]$ so (2.4) turns to

$$\phi(\xi) = \frac{1+2e^{\tau}}{1+e^{\tau}}. \tag{2.63}$$

By using the **Mathematica** software, we obtain a polynomial and. By setting the coefficients of this polynomial corresponding to the same powers to zero, we derive a system of algebraic equations for a_0, a_1, a_2, ω .

$$\begin{aligned}
&a_0^2k^2 + a_1^2k^2 + a_2^2k^2 + 2a_0k^2 + 2a_0a_1k^2 + 2a_1k^2 + 2a_0a_2k^2 + 2a_1a_2k^2 + 2a_2k^2 - 2a_0\omega^2 - 2a_1\omega^2 - 2a_2\omega^2 = 0,\\
&2a_1k^2\omega^2 + 4a_2k^2\omega^2 + 4a_0^2k^2 + 6a_1^2k^2 + 8a_2^2k^2 + 8a_0k^2 + 10a_0a_1k^2\\
&+ 10a_1k^2 + 12a_0a_2k^2 + 14a_1a_2k^2 + 12a_2k^2 - 8a_0\omega^2 - 10a_1\omega^2 - 12a_2\omega^2 = 0,\\
&a_0^2k^2 + 4a_1^2k^2 + 16a_2^2k^2 + 2a_0k^2 + 4a_0a_1k^2 + 4a_1k^2 + 8a_0a_2k^2 + 16a_1a_2k^2\\
&+ 8a_2k^2 - 2a_0\omega^2 - 4a_1\omega^2 - 8a_2\omega^2 = 0,\\
&8a_2k^2\omega^2 + 6a_0^2k^2 + 13a_1^2k^2 + 24a_2^2k^2 + 12a_0k^2 + 18a_0a_1k^2 + 18a_1k^2\\
&+ 26a_0a_2k^2 + 36a_1a_2k^2 + 26a_2k^2 - 12a_0\omega^2 - 18a_1\omega^2 - 26a_2\omega^2 = 0,\\
&- 2a_1k^2\omega^2 - 8a_2k^2\omega^2 + 4a_0^2k^2 + 12a_1^2k^2 + 32a_2^2k^2 + 8a_0k^2 + 14a_0a_1k^2\\
&+ 14a_1k^2 + 24a_0a_2k^2 + 40a_1a_2k^2 + 24a_2k^2 - 8a_0\omega^2 - 14a_1\omega^2 - 24a_2\omega^2 = 0.
\end{aligned} \tag{2.64}$$

After solving the resulting algebraic system and with help of **Mathematica**, we obtain the followin results:

Case 1:

$$\left\{a_0 \to \frac{24k^2}{k^2-1}, a_1 \to -\frac{36k^2}{k^2-1}, a_2 \to \frac{12k^2}{k^2-1}, \omega \to \pm\frac{k}{\sqrt{1-k^2}}\right\} \tag{2.65}$$

Putting these results in Eq (2.3) and (2.63) leads to

$$U(\xi) = -\frac{6k^2}{(k^2-1)(\cosh(\xi)+1)} \tag{2.66}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = -\frac{3k^2\operatorname{sech}^2\left(\frac{1}{2}k\left(\frac{t}{\sqrt{1-k^2}} \mp x\right)\right)}{k^2-1} \tag{2.67}$$

where $\xi = kx - \omega t$

Case 2:

$$\left\{a_0 \to -\frac{26k^2}{k^2+1}, a_1 \to \frac{36k^2}{k^2+1}, a_2 \to -\frac{12k^2}{k^2+1}, \omega \to \pm\frac{k}{\sqrt{k^2+1}}\right\} \tag{2.68}$$

Putting these results in Eq (2.3) and (2.63) leads to

$$U(\xi) = -\frac{2k^2(\cosh(\xi)-2)}{(k^2+1)(\cosh(\xi)+1)} \tag{2.69}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{k^2\left(3\operatorname{sech}^2\left(\frac{1}{2}k\left(\frac{t}{\sqrt{k^2+1}} \mp x\right)\right)-2\right)}{k^2+1} \tag{2.70}$$

where $\xi = kx - \omega t$

Group 7: We obtain $p = [-1-i, 1-i, -1, 1]$ and $q = [i, -i, i, -i]$ so (2.4) turns to

$$\phi(\xi) = \cot(\xi) + 1. \tag{2.71}$$

By using the **Mathematica** software, we obtain a polynomial . By setting the coefficients of this polynomial corresponding to the same powers to zero, we derive a system of algebraic equations for a_0, a_1, a_2, ω .

$$
\begin{aligned}
&-2a_2k^2\omega^2-\frac{1}{2}a_0^2k^2-\frac{1}{2}a_1^2k^2-\frac{1}{2}a_2^2k^2-a_0a_1k^2-a_1k^2\\
&-a_0a_2k^2-a_1a_2k^2-a_2k^2-a_0k^2+a_0\omega^2+a_1\omega^2+a_2\omega^2=0,\\
&-8a_2k^2\omega^2-\frac{1}{2}a_1^2k^2-3a_2^2k^2-a_0a_2k^2-3a_1a_2k^2-a_2k^2+a_2\omega^2=0,\\
&-2a_1k^2\omega^2-4a_2k^2\omega^2-a_1^2k^2-2a_2^2k^2-a_0a_1k^2-2a_0a_2k^2-3a_1a_2k^2-2a_2k^2-a_1k^2+a_1\omega^2+2a_2\omega^2=0,\\
&-2a_1k^2\omega^2-4a_2k^2\omega^2-2a_2^2k^2-a_1a_2k^2=0,\\
&-6a_2k^2\omega^2-\frac{1}{2}a_2^2k^2=0.
\end{aligned}
\tag{2.72}
$$

After solving the resulting algebraic system and with help of **Mathematica**, we obtain the following results:

Case 1:

$$\left\{a_0\to\frac{16k^2}{4k^2-1},a_1\to-\frac{24k^2}{4k^2-1},a_2\to\frac{12k^2}{4k^2-1},\omega\to\pm\frac{k}{\sqrt{1-4k^2}}\right\}\tag{2.73}$$

Putting these results in Eq (2.3) and (2.71) leads to

$$U(\xi)=\frac{4k^2(\cos(2\xi)+2)\csc^2(\xi)}{4k^2-1}\tag{2.74}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t)=\frac{4k^2\left(3\csc^2\left(k\left(\frac{t}{\sqrt{1-4k^2}}\mp x\right)\right)-2\right)}{4k^2-1}\tag{2.75}$$

where $\xi=kx-\omega t$

Case 2:

$$\left\{a_0\to-\frac{24k^2}{4k^2+1},a_1\to\frac{24k^2}{4k^2+1},a_2\to-\frac{12k^2}{4k^2+1},\omega\to\pm\frac{k}{\sqrt{4k^2+1}}\right\}\tag{2.76}$$

Putting these results in Eq (2.3) and (2.71) leads to

$$U(\xi)=-\frac{12k^2\csc^2(\xi)}{4k^2+1}\tag{2.77}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = -\frac{12k^2\csc^2\left(k\left(\frac{t}{\sqrt{4k^2+1}} \mp x\right)\right)}{4k^2+1} \tag{2.78}$$

where $\xi = kx - \omega t$

Case 3:

$$\left\{a_0 \to -3, a_1 \to 3, a_2 \to -\frac{3}{2}, k \to -\frac{1}{2}, \omega \to \pm\frac{1}{2\sqrt{2}}\right\} \tag{2.79}$$

Putting these results in Eq (2.3) and (2.71) leads to

$$U(\xi) = \frac{1}{2}(-3)\csc^2(\xi) \tag{2.80}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{3}{\cos\left(\frac{t}{\sqrt{2}} \mp x\right) - 1} \tag{2.81}$$

where $\xi = kx - \omega t$

Case 4:

$$\left\{a_0 \to -3, a_1 \to 3, a_2 \to -\frac{3}{2}, k \to \frac{1}{2}, \omega \to \pm\frac{1}{2\sqrt{2}}\right\} \tag{2.82}$$

Putting these results in Eq (2.3) and (2.71) leads to

$$U(\xi) = \frac{1}{2}(-3)\csc^2(\xi) \tag{2.83}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = \frac{3}{\cos\left(\frac{t}{\sqrt{2}} \mp x\right) - 1} \tag{2.84}$$

where $\xi = kx - \omega t$

Case 5:

$$\left\{a_0 \to 2, a_1 \to -3, a_2 \to \frac{3}{2}, k \to -\frac{i}{2}, \omega \to \pm\frac{i}{2\sqrt{2}}\right\} \tag{2.85}$$

Putting these results in Eq (2.3) and (2.71) leads to

$$U(\xi) = \frac{3\csc^2(\xi)}{2} - 1 \tag{2.86}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = -\frac{3}{\cosh\left(\frac{t}{\sqrt{2}} \mp x\right) - 1} - 1 \tag{2.87}$$

where $\xi = kx - \omega t$

Case 6:

$$\left\{a_0 \to 2, a_1 \to -3, a_2 \to \frac{3}{2}, k \to \frac{i}{2}, \omega \to \pm\frac{i}{2\sqrt{2}}\right\} \tag{2.88}$$

Putting these results in Eq (2.3) and (2.71) leads to

$$U(\xi) = \frac{3\csc^2(\xi)}{2} - 1 \tag{2.89}$$

Thus, the propagating solution of the PDE given by (2.1) can be achieved as

$$u(x,t) = -\frac{3}{\cosh\left(\frac{t}{\sqrt{2}} \mp x\right) - 1} - 1 \tag{2.90}$$

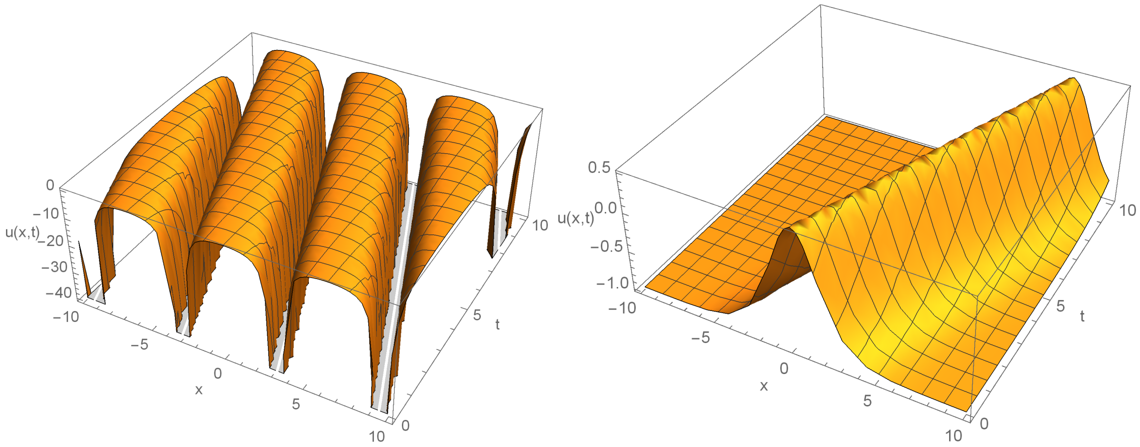
where $\xi = kx - \omega t$

Group 8: We obtain $p = [-2 - i, 2 - i, -1, 1]$ and $q = [i, -i, i, -i]$ so (2.4) turns to

$$\phi(\xi) = \cot(\xi) + 2. \tag{2.91}$$

Group 9: We obtain $p = [1 - i, -1 - i, -1, 1]$ and $q = [i, -i, i, -i]$ so (2.4) turns to

$$\phi(\xi) = \cot(\xi) - 1. \tag{2.92}$$



(a) (b)

Figure 2.1: 3D Plot (a), (b) of some exact solutions of (2.1) given by (2.9) and (2.58).

Chapter 3

Application of generalized Riccati mapping method to the long-short wave interaction equation

The long-short wave interaction equation (LSWIE) [26, 27, 28, 29, 30] is a mathematical model that describes the nonlinear interaction between a long wave and a short wave. This type of interaction is common in fluid dynamics, plasma physics, nonlinear optics, and other areas where waves of different scales coexist and influence each other. General Form of the Long-Short Wave Interaction Equations One common form of the LSWIE system is:

$$\begin{aligned} &i\frac{\partial u(x,t)}{\partial t}+\frac{\partial^2 u(x,t)}{\partial x^2}-u(x,t)v(x,t)=0,\\ &\frac{\partial v(x,t)}{\partial t}+\frac{\partial v(x,t)}{\partial x}+\frac{\partial\left|u(x,t)\right|^2}{\partial x}=0. \end{aligned} \tag{3.1}$$

We will proceed to solve this equation using the generalized Riccati mapping method. Applying the traveling wave transformation where

$$\begin{aligned} &\xi=kx-t\omega,\\ &\theta=\rho t+\eta x,\\ &u(x,t)=U(\xi)=e^{i\theta}U(\xi)=e^{i(\rho t+\eta x)}U(kx-t\omega),\\ &v(x,t)=V(\xi)=V(kx-t\omega) \end{aligned} \tag{3.2}$$

to reduce Eq (3.1) into the following NLODEs:

$$k^2U''(\xi) - \frac{kU(\xi)^3}{\omega - k} - \left(\eta^2 + \rho\right)U(\xi) = 0 \tag{3.3}$$

$$V(\xi) = \frac{kU(\xi)^2}{\omega - k} \tag{3.4}$$

where $\omega = 2\eta k$.

By balancing U'' with U^3 we have $N = 1$ Hence the formal solution of Eq (3.3) takes the form:

$$U(\xi) = \frac{a_{-1}}{Q(\xi)} + a_1Q(\xi) + a_0 \tag{3.5}$$

Where

$$Q'(\xi) = pQ(\xi) + qQ(\xi)^2 + r \tag{3.6}$$

Proceeding as indicate in the method, we get a system of nonlinear algebric equations

$$\begin{aligned}
&-a_0\eta^2 + a_{-1}k^2pq + a_1k^2pr - \frac{a_0^3k}{\omega - k} - \frac{6a_{-1}a_1a_0k}{\omega - k} - a_0\rho = 0,\\
&+2a_{-1}k^2r^2 - \frac{a_{-1}^3k}{\omega - k} = 0, +3a_{-1}k^2pr - \frac{3a_{-1}^2a_0k}{\omega - k} = 0,\\
&-a_{-1}\eta^2 + a_{-1} + k^2p^2 + 2a_{-1}k^2qr - \frac{3a_{-1}a_0^2k}{\omega - k} - \frac{3a_{-1}^2a_1k}{\omega - k} - a_{-1}\rho = 0,\\
&-a_1\eta^2 + a_1k^2p^2 + 2a_1k^2qr - \frac{3a_{-1}a_1^2k}{\omega - k} - \frac{3a_0^2a_1k}{\omega - k} - a_1\rho = 0,\\
&3a_1k^2pq - \frac{3a_0a_1^2k}{\omega - k} = 0, 2a_1k^2q^2 - \frac{a_1^3k}{\omega - k} = 0
\end{aligned} \tag{3.7}$$

The resulting algebraic system Eq.(3.7)is solved with the help of **Mathematica** to determine the values of the unkown constants $a_0, a_1, a_{-1}, k, \omega, \rho, \eta$

$$\begin{aligned}
&\left\{a_0 \to -\frac{\sqrt{k}p\sqrt{\omega - k}}{\sqrt{2}}, a_1 \to 0, a_{-1} \to -\sqrt{2}\sqrt{kr}\sqrt{\omega - k}, \rho \to -\eta^2 - \frac{1}{2}k^2\left(p^2 - 4qr\right)\right\},\\
&\left\{a_0 \to -\frac{\sqrt{k}p\sqrt{\omega - k}}{\sqrt{2}}, a_1 \to -\sqrt{2}\sqrt{kq}\sqrt{\omega - k}, a_{-1} \to 0, \rho \to -\eta^2 - \frac{1}{2}k^2\left(p^2 - 4qr\right)\right\},\\
&\left\{a_0 \to \frac{\sqrt{k}p\sqrt{\omega - k}}{\sqrt{2}}, a_1 \to 0, a_{-1} \to \sqrt{2}\sqrt{kr}\sqrt{\omega - k}, \rho \to -\eta^2 - \frac{1}{2}k^2\left(p^2 - 4qr\right)\right\},\\
&\left\{a_0 \to \frac{\sqrt{k}p\sqrt{\omega - k}}{\sqrt{2}}, a_1 \to \sqrt{2}\sqrt{kq}\sqrt{\omega - k}, a_{-1} \to 0, \rho \to -\eta^2 - \frac{1}{2}k^2\left(p^2 - 4qr\right)\right\}.
\end{aligned} \tag{3.8}$$

Insertion Eq.(3.8) with the aid of Equation (3.6) into equation (3.5) we get the following system of algebraic equation:

Type 1:$\Delta = p^2 - 4qr > 0, pq \neq 0, pr \neq 0$:

$$u_{1,1}(x,t) = \left(\pm \frac{\sqrt{k}p\sqrt{\omega - k}}{\sqrt{2}} \mp \frac{\sqrt{2}\sqrt{kr}\sqrt{\omega - k}}{\frac{\frac{1}{2}\sqrt{p^2-4qr}\tanh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)+p}{2q}} \right) \times \exp\left(i\left(-\frac{1}{2}k^2\left(p^2 - 4qr\right) - \eta^2 t + \eta x \right) \right), \tag{3.9}$$

$$u_{1,2}(x,t) = \left(\pm \frac{\sqrt{k}p\sqrt{\omega - k}}{\sqrt{2}} \mp \frac{\sqrt{2\sqrt{kr}\sqrt{\omega - k}}\left(\frac{1}{2}\sqrt{p^2 - 4qr}\tanh\left(\frac{1}{2}\sqrt{p^2 - 4qr}(kx - t\omega)\right) + p\right)}{2q} \right) \times \exp\left(i\left(-\frac{1}{2}k^2\left(p^2 - 4qr\right) - \eta^2 t + \eta x \right) \right), \tag{3.10}$$

$$u_{2,1}(x,t) = \left(\pm \frac{\sqrt{k}p\sqrt{\omega - k}}{\sqrt{2}} \mp \frac{\sqrt{2}\sqrt{kr}\sqrt{\omega - k}}{\frac{\sqrt{p^2 - 4qr}\coth\left(\frac{1}{2}\sqrt{p^2 - 4qr}(kx - t\omega)\right) + p}{2q}} \right) \times \exp\left(i\left(-\frac{1}{2}k^2(p^2 - 4qr) - \eta^2 t + \eta x \right) \right) \tag{3.11}$$

$$u_{2,2}(x,t) = \left(\pm \frac{\sqrt{k}p\sqrt{\omega - k}}{\sqrt{2}} \mp \frac{\left(\sqrt{2}\sqrt{kr}\sqrt{\omega - k}\right)\left(\sqrt{p^2 - 4qr}\coth\left(\frac{1}{2}\sqrt{p^2 - 4qr}(kx - t\omega)\right) + p\right)}{2q} \right) \times \exp\left(i\left(-\frac{1}{2}k^2\left(p^2 - 4qr\right) - \eta^2 t + \eta x \right) \right) \tag{3.12}$$

$$u_{3,1}(x,t) = \left(\pm \frac{\sqrt{k}p\sqrt{\omega - k}}{\sqrt{2}} \mp \frac{\sqrt{2}\sqrt{kr}\sqrt{\omega - k}}{\frac{p+\sqrt{p^2-4qr}\left(\tanh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\pm i\operatorname{sech}\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\right)}{2q}} \right) \times \exp\left(i\left(t\left(-\eta^2 - \frac{1}{2}k^2\left(p^2 - 4qr\right) \right) + \eta x \right) \right) \tag{3.13}$$

$$u_{3,2}(x,t)=\left\{\begin{matrix}\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \mp\frac{\left(\sqrt{2}\sqrt{k}q\sqrt{\omega-k}\right)\left(p+\sqrt{p^2-4qr}\left(\tanh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\pm i\operatorname{sech}\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\right)\right)}{2q}\end{matrix}\right\}$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.14}$$

$$u_{4,1}(x,t)=\left(\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\mp\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{\sqrt{p^2-4qr}\left(\coth\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\pm\operatorname{csch}\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\right)+p}{2q}}\right)$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.15}$$

$$u_{4,2}(x,t)=\left\{\begin{matrix}\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \mp\frac{\left(\sqrt{2}\sqrt{k}q\sqrt{\omega-k}\right)\left(\sqrt{p^2-4qr}\left(\coth\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\pm\operatorname{csch}\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\right)+p\right)}{2q}\end{matrix}\right\}$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.16}$$

$$u_{5,1}(x,t)=\left(\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\mp\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{\sqrt{p^2-4qr}\left(\tanh\left(\frac{1}{4}\sqrt{p^2-4qr}(kx-t\omega)\right)\pm\coth\left(\frac{1}{4}\sqrt{p^2-4qr}(kx-t\omega)\right)\right)+2p}{4q}}\right)$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.17}$$

$$u_{5,2}(x,t)=\left\{\begin{matrix}\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \mp\frac{\left(\sqrt{2}\sqrt{k}q\sqrt{\omega-k}\right)\left(\sqrt{p^2-4qr}\left(\tanh\left(\frac{1}{4}\sqrt{p^2-4qr}(kx-t\omega)\right)\pm\coth\left(\frac{1}{4}\sqrt{p^2-4qr}(kx-t\omega)\right)\right)+2p\right)}{4q}\end{matrix}\right\}$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.18}$$

$$u_{6,1}(x,t)=\left(\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\pm\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{\frac{\sqrt{\left(A^2+B^2\right)\left(p^2-4qr\right)}-A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)}{A\sinh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)+B}-p}{2q}}\right)$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.19}$$

$$\begin{aligned} u_{6,2}(x,t) = & \left\{ \pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \pm \frac{\left(\sqrt{2}\sqrt{k}q\sqrt{\omega-k}\right)\left(\frac{\sqrt{\left(A^2+B^2\right)\left(p^2-4qr\right)}-A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)}{A\sinh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)+B}-p\right)}{2q} \right\} \\ & \times exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.20}$$

$$\begin{aligned} u_{7,1}(x,t) = & \left\{ \pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \mp -\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{\frac{\sqrt{\left(B^2-A^2\right)\left(p^2-4qr\right)}+A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)}{A\sinh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)+B}+p}{2q}} \right\} \\ & \times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.21}$$

$$\begin{aligned} u_{7,2}(x,t) = & \left\{ \pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \mp \frac{\left(\sqrt{2}\sqrt{k}q\sqrt{\omega-k}\right)\left(\frac{\sqrt{\left(B^2-A^2\right)\left(p^2-4qr\right)}+A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)}{A\sinh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)+B}+p\right)}{2q} \right\} \\ & \times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.22}$$

$-A^2+A$ and are B are two non - zero real constants satisfying $B^2-A^2>0$

$$\begin{aligned} u_{8,1}(x,t) = & \left\{ \pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \pm \frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{2r\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)}{\sqrt{p^2-4qr}\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)-p\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)}} \right\} \\ & \times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.23}$$

$$\begin{aligned} u_{8,2}(x,t) = & \left\{ \pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \pm \frac{\sqrt{2}2\sqrt{k}qr\sqrt{\omega-k}\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)}{\sqrt{p^2-4qr}\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)-p\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)} \right\} \\ & \times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.24}$$

$$\begin{aligned} u_{9,1}(x,t) = & \left\{ \pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \mp \frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{2r\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)}{p\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)-\sqrt{p^2-4qr}\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)}} \right\} \\ & \times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.25}$$

$$\begin{aligned} u_{9,2}(x,t) = & \left\{ \pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \mp \frac{\sqrt{2}2\sqrt{k}qr\sqrt{\omega-k}\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)}{p\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)-\sqrt{p^2-4qr}\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)} \right\} \\ & \times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.26}$$

$$\begin{aligned} u_{10,1}(x,t) = &\left\{ \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \pm \frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{2r\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)}{\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)-\left(p\cosh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\pm i\sqrt{p^2-4qr}\right)}} \right\} \\ &\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.27}$$

$$\begin{aligned} u_{10,2}(x,t) = &\left\{ \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \pm \frac{\sqrt{2}2\sqrt{k}qr\sqrt{\omega-k}\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)}{\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)-\left(p\cosh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\pm i\sqrt{p^2-4qr}\right)} \right\} \\ &\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.28}$$

$$\begin{aligned} u_{11,1}(x,t) = &\left\{ \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \pm \frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{2r\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)}{\left(\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\pm\sqrt{p^2-4qr}\right)-p\sinh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)}} \right\} \\ &\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.29}$$

$$\begin{aligned} u_{11,2}(x,t) = &\left\{ \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \pm \frac{\sqrt{2}2\sqrt{k}qr\sqrt{\omega-k}\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)}{\left(\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)\pm\sqrt{p^2-4qr}\right)-p\sinh\left(\sqrt{p^2-4qr}(kx-t\omega)\right)} \right\} \\ &\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.30}$$

$$\begin{aligned} u_{12,1}(x,t) = &\left\{ \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \pm \frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{4r\sinh\left(\frac{1}{24}\sqrt{p^2-4qr}(kx-t\omega)\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}(kx-t\omega)\right)}{2\sqrt{p^2-4qr}\cosh^2\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)-2p\sinh\left(\frac{1}{24}\sqrt{p^2-4qr}(kx-t\omega)\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}(kx-t\omega)\right)-\sqrt{p^2-4qr}}} \right\} \\ &\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.31}$$

$$\begin{aligned} u_{12,2}(x,t) = &\left\{ \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \pm \frac{\sqrt{2}4\sqrt{k}qr\sqrt{\omega-k}\sinh\left(\frac{1}{24}\sqrt{p^2-4qr}(kx-t\omega)\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}(kx-t\omega)\right)}{2\sqrt{p^2-4qr}\cosh^2\left(\frac{1}{2}\sqrt{p^2-4qr}(kx-t\omega)\right)-2p\sinh\left(\frac{1}{24}\sqrt{p^2-4qr}(kx-t\omega)\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}(kx-t\omega)\right)-\sqrt{p^2-4qr}} \right\} \\ &\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \end{aligned} \tag{3.32}$$

Type 2:$\Delta = p^2 - 4qr < 0, pq \neq 0, pr \neq 0$

$$u_{13,1}(x,t)=\left\{\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\pm\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{1}{2q}\left(\sqrt{-\left(p^2-4qr\right)}\tan\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)-p\right)}\right\}$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.33}$$

$$u_{13,2}(x,t)=\left\{\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\pm\frac{\sqrt{2}\sqrt{k}q\sqrt{\omega-k}}{2q}\left(\sqrt{-\left(p^2-4qr\right)}\tan\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)-p\right)\right\}$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.34}$$

$$u_{14,1}(x,t)=\left\{\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\mp\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{1}{2q}\left(\sqrt{-\left(p^2-4qr\right)}\cot\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)+p\right)}\right\}$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.35}$$

$$u_{14,2}(x,t)=$$
$$\left\{\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\mp\frac{\sqrt{2}\sqrt{k}q\sqrt{\omega-k}}{2q}\left(\sqrt{-\left(p^2-4qr\right)}\cot\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)+p\right)\right\}$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.36}$$

$$u_{15,1}(x,t)=\left\{\begin{array}{c}\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \pm\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{1}{2q}\left(\sqrt{-\left(p^2-4qr\right)}\left(\tan\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\pm\sec\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\right)-p\right)}\end{array}\right\}$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.37}$$

$$u_{15,2}(x,t)=$$
$$\left\{\begin{array}{c}\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \pm\frac{\sqrt{2}\sqrt{k}q\sqrt{\omega-k}}{2q}\left(\sqrt{-\left(p^2-4qr\right)}\left(\tan\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\pm\sec\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\right)-p\right)\end{array}\right\}$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.38}$$

$$u_{16,1}(x,t)=\left\{\begin{array}{c}\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \mp\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{1}{2q}\left(\sqrt{-\left(p^2-4qr\right)}\left(\cot\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\pm\csc\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\right)+p\right)}\end{array}\right\}$$
$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.39}$$

$$
\begin{aligned}
&u_{16,2}(x,t) = \\
&\left\{\begin{array}{c} \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \\ \mp \frac{\sqrt{2}\sqrt{k}q\sqrt{\omega-k}}{2q}\left(\sqrt{-\left(p^2-4qr\right)}\left(\cot\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right) \pm \csc\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\right)+p\right) \end{array}\right\} \\
&\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right)
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
u_{17,1}(x,t) = &\left\{\begin{array}{c} \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \\ \pm \frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{1}{4q}\left(\sqrt{-\left(p^2-4qr\right)}\left(\tan\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)-\cot\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\right)-2p\right)} \end{array}\right\} \\
&\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right)
\end{aligned}
\tag{3.41}
$$

$$
\begin{aligned}
&u_{17,2}(x,t) = \\
&\left\{\begin{array}{c} \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \\ \pm \frac{\sqrt{2}\sqrt{k}q\sqrt{\omega-k}}{4q}\left(\sqrt{-\left(p^2-4qr\right)}\left(\tan\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)-\cot\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\right)-2p\right) \end{array}\right\} \\
&\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right)
\end{aligned}
\tag{3.42}
$$

$$
\begin{aligned}
u_{18,1}(x,t) = &\left\{\begin{array}{c} \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \\ \pm \frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{1}{2q}\left(\frac{\pm\sqrt{\left(A^2-B^2\right)\left(-\left(p^2-4qr\right)\right)}-A\sqrt{-\left(p^2-4qr\right)}\cos\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{A\sinh\left(-\left(p^2-4qr\right)(kx-t\omega)\right)+B}-p\right)} \end{array}\right\} \\
&\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right)
\end{aligned}
\tag{3.43}
$$

$$
\begin{aligned}
u_{18,2}(x,t) = &\left\{\begin{array}{c} \pm \frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}} \\ \pm \frac{\sqrt{2}\sqrt{k}q\sqrt{\omega-k}}{2q}\left(\frac{\pm\sqrt{\left(A^2-B^2\right)\left(-\left(p^2-4qr\right)\right)}-A\sqrt{-\left(p^2-4qr\right)}\cos\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{A\sinh\left(-\left(p^2-4qr\right)(kx-t\omega)\right)+B}-p\right) \end{array}\right\} \\
&\times \exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right)
\end{aligned}
\tag{3.44}
$$

$$u_{19,1}(x,t)=\left\{\begin{array}{c}\pm\dfrac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \mp\dfrac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{1}{2q}\left(\frac{\pm\sqrt{\left(A^2-B^2\right)\left(-\left(p^2-4qr\right)\right)}-A\sqrt{-\left(p^2-4qr\right)}\sin\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{A\sinh\left(-\left(p^2-4qr\right)(kx-t\omega)\right)+B}-p\right)}\end{array}\right\} \times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.45}$$

$$u_{19,2}(x,t)=\left\{\begin{array}{c}\pm\dfrac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \mp\dfrac{\sqrt{2}\sqrt{k}q\sqrt{\omega-k}}{2q}\left(\dfrac{\pm\sqrt{\left(A^2-B^2\right)\left(-\left(p^2-4qr\right)\right)}-A\sqrt{-\left(p^2-4qr\right)}\sin\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{A\sinh\left(-\left(p^2-4qr\right)(kx-t\omega)\right)+B}-p\right)\end{array}\right\} \times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.46}$$

$-A^2+A$ and are B are two non - zero real constants satisfying $B^2-A^2>0$

$$u_{20,1}(x,t)=\left\{\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\mp\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{2r\cos\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{\sqrt{-\left(p^2-4qr\right)}\sin\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)+p\cos\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}}\right\} \times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.47}$$

$$u_{20,2}(x,t)=\left\{\begin{array}{c}\pm\dfrac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \mp\dfrac{\sqrt{2}2\sqrt{k}qr\sqrt{\omega-k}\cos\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{\sqrt{-\left(p^2-4qr\right)}\sin\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)+p\cos\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}\end{array}\right\} \times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.48}$$

$$u_{21,1}(x,t)=\left\{\begin{array}{c}\pm\dfrac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \pm\dfrac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{2r\sin\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{\sqrt{-\left(p^2-4qr\right)}\cos\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)-p\sin\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}}\end{array}\right\} \times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.49}$$

$$u_{21,2}(x,t)=\left\{\begin{array}{c}\pm\dfrac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \pm\dfrac{\sqrt{2}2\sqrt{k}qr\sqrt{\omega-k}\sin\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{\sqrt{-\left(p^2-4qr\right)}\cos\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)-p\sin\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}\end{array}\right\} \times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right) \tag{3.50}$$

$$u_{22,1}(x,t)=\left\{\begin{array}{c}\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \mp\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{2r\cos\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{\sqrt{-\left(p^2-4qr\right)}\sin\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)+\left(p\cos\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\pm\sqrt{-\left(p^2-4qr\right)}\right)}}\end{array}\right\}$$

$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right)\tag{3.51}$$

$$u_{22,2}(x,t)=\left\{\begin{array}{c}\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \mp\frac{\sqrt{2}2\sqrt{k}qr\sqrt{\omega-k}\cos\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{\sqrt{-\left(p^2-4qr\right)}\sin\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)+\left(p\cos\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\pm\sqrt{-\left(p^2-4qr\right)}\right)}\end{array}\right\}$$

$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right)\tag{3.52}$$

$$u_{23,1}(x,t)=\left(\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\pm\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{2r\sin\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{\left(\sqrt{-\left(p^2-4qr\right)}\cos\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\pm\sqrt{-\left(p^2-4qr\right)}\right)-p\sin\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}}\right)$$

$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right)\tag{3.53}$$

$$u_{23,2}(x,t)=\left\{\begin{array}{c}\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \pm\frac{\sqrt{2}2\sqrt{k}qr\sqrt{\omega-k}\sin\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{\left(\sqrt{-\left(p^2-4qr\right)}\cos\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\pm\sqrt{-\left(p^2-4qr\right)}\right)-p\sin\left(\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}\end{array}\right\}$$

$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right)\tag{3.54}$$

$$u_{24,1}(x,t)=\left\{\begin{array}{c}\pm\frac{\sqrt{k}p\sqrt{\omega-k}}{\sqrt{2}}\\ \pm\frac{\sqrt{2}\sqrt{k}r\sqrt{\omega-k}}{\frac{4r\sin\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\cos\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)}{2\sqrt{-\left(p^2-4qr\right)}\mathrm{Cos}^2\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)-2p\sin\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\cos\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)-\sqrt{-\left(p^2-4qr\right)}}}\end{array}\right\}$$

$$\times\exp\left(i\left(t\left(-\eta^2-\frac{1}{2}k^2\left(p^2-4qr\right)\right)+\eta x\right)\right)\tag{3.55}$$

$$
\begin{aligned}
u_{24,2}(x,t) = & \left\{ \begin{array}{c} \pm \frac{\sqrt{k} p \sqrt{\omega - k}}{\sqrt{2}} \\ \pm \frac{\sqrt{2}\sqrt{k} q \sqrt{\omega - k}\left(4r \sin\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\cos\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\right)}{2\sqrt{-\left(p^2-4qr\right)}\mathrm{Cos}^2\left(\frac{1}{2}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)-2p\sin\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)\cos\left(\frac{1}{4}\sqrt{-\left(p^2-4qr\right)}(kx-t\omega)\right)-\sqrt{-\left(p^2-4qr\right)}} \end{array} \right\} \\
& \times \exp\left(i\left(t\left(-\eta^2 - \frac{1}{2}k^2\left(p^2-4qr\right)\right) + \eta x\right)\right)
\end{aligned}
\tag{3.56}
$$

Type 3:$r = 0, pq \neq 0$ we have:

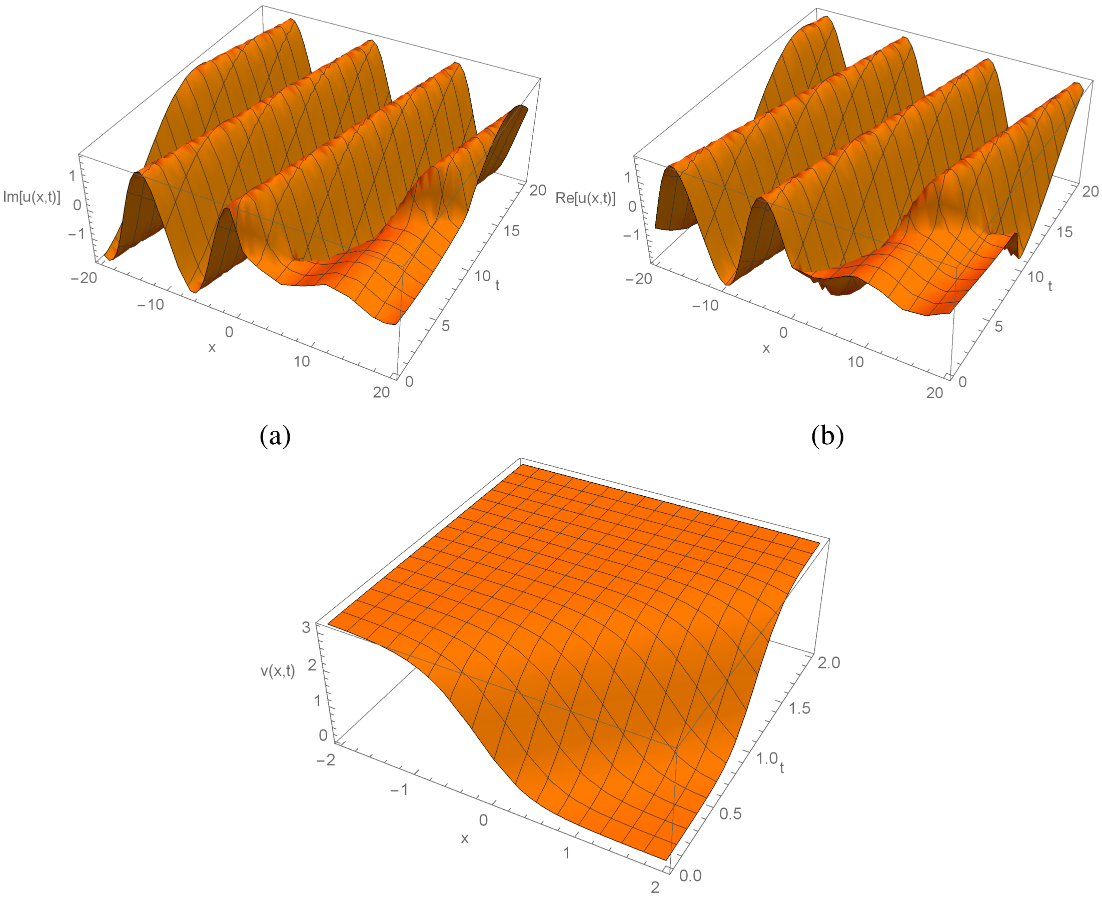
$$
u_{25,1}(x,t) = \pm \frac{\sqrt{k} p \sqrt{\omega - k}}{\sqrt{2}} \exp\left(i\left(-\eta^2 - \frac{1}{2}k^2 p^2\right)t + \eta x\right) \tag{3.57}
$$

$$
\begin{aligned}
u_{25,2}(x,t) = & \left(\pm \frac{\sqrt{k} p \sqrt{\omega - k}}{\sqrt{2}} \mp \frac{\sqrt{2} d \sqrt{k} p q \sqrt{\omega - k}}{q(d - \sinh(p(kx - t\omega)) + \cosh(p(kx - t\omega)))}\right) \\
& \times \exp\left(i\left(-\eta^2 - \frac{1}{2}k^2 p^2\right)t + \eta x\right)
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
u_{26,2}(x,t) = & \left(\pm \frac{\sqrt{k} p \sqrt{\omega - k}}{\sqrt{2}} \mp \frac{\sqrt{2}\sqrt{k} q \sqrt{\omega - k} p(\sinh(p(kx - t\omega)) + \cosh(p(kx - t\omega)))}{q(d + \sinh(p(kx - t\omega)) + \cosh(p(kx - t\omega)))}\right) \\
& \times \exp\left(i\left(-\eta^2 - \frac{1}{2}k^2 p^2\right)t + \eta x\right)
\end{aligned}
\tag{3.59}
$$

Type4 :when $r = p = 0, q \neq 0$ we have no nontrivial solutions

Remark 3.0.1. *We substitute the value of* $u_{i,j}(x,t)$ *for each case into Eq* (3.4) *to obtain the value of* $v_{i,j}(x,t)$

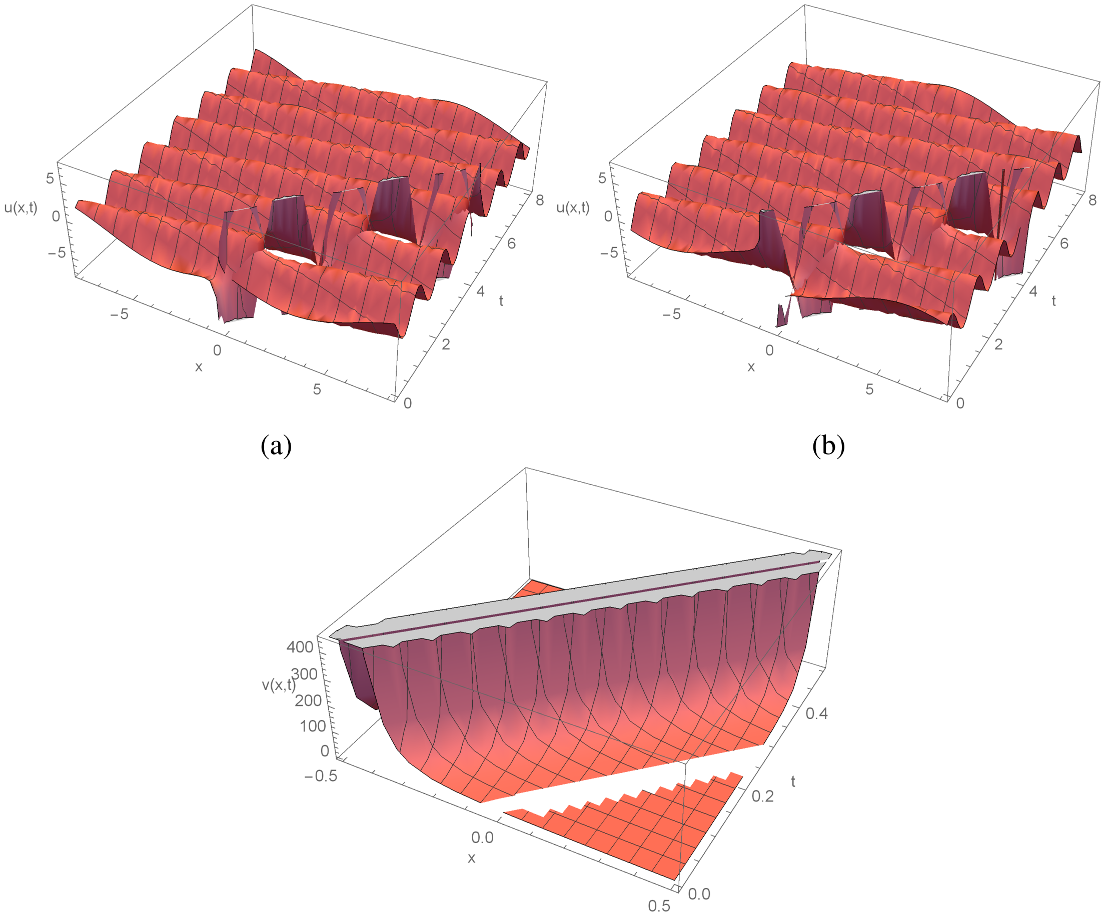


(a)

(b)

(c)

Figure 3.1: 3D Plot (a), (b) and (c) of some exact solutions of (3.1) given by (3.10).



(a)

(b)

(c)

Figure 3.2: 3D Plot (a), (b) and (c) of some exact solutions of (3.1) given by (3.33).

Chapter 4

Application of the Modified Exponential Function Method to the coupled Klein-Gordon equation

The coupled Klein-Gordon equations [15, 16, 32, 24, 24, 19, 22, 22, 23, 25] refer to a system of partial differential equations (PDEs) that generalize the standard Klein-Gordon equation to include multiple interacting fields. These equations often arise in theoretical physics, especially in quantum field theory, nonlinear optics, and condensed matter physics In the form:

$$-\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial^2 u(x,t)}{\partial x^2}+2u(x,t)v(x,t)-u(x,t)+2u(x,t)^3=0, \tag{4.1}$$

$$-\frac{\partial v(x,t)}{\partial t}+\frac{\partial v(x,t)}{\partial x}-4u(x,t)\frac{\partial u(x,t)}{\partial t}=0. \tag{4.2}$$

We will proceed to solve this equation by using of the Modified Exponential Function Method. We consider the following transformation

$$\xi = kx - t\omega, \tag{4.3}$$

$$u(x,t) = U(\xi) = U(kx - t\omega), \tag{4.4}$$

$$v(x,t) = V(\xi) = V(kx - t\omega). \tag{4.5}$$

Where k and ω are a constants

Then we Using transformation Eq (4.3), then transformed into the following ordinary differential equation

$$k^2U''(\xi) - \omega^2U''(\xi) + 2U(\xi)^3 - U(\xi) + 2U(\xi)V(\xi) = 0 \tag{4.6}$$

$$V(\xi) = -\frac{\omega U^2(\xi)}{2(k+\omega)}. \tag{4.7}$$

Balancing now the highest-order derivative U^3 and the nonlinear term U'' we obtain $3(N-M) = (N-M)+2$ which simplifies to $N-M=1$. If we set $M=1$ this yields $N=1$ meaning the solution takes the form:

$$U(\xi) = \frac{a_1e^{-\Phi(\xi)} + a_2e^{-2\Phi(\xi)} + a_0}{b_1e^{-\Phi(\xi)} + b_0} \tag{4.8}$$

By using the **Mathematica** software, we obtain a polynomial in $e^n\Phi(\xi)$. By setting the coefficients of this polynomial corresponding to the same powers of $e^n\Phi(\xi)$ to zero, we derive a system of algebraic equations for a_0, a_1, a_2, ω .

$$\begin{aligned}
&-2a_2k^2\omega^2 - \frac{1}{2}a_0^2k^2 - a_0k^2 + a_0\omega^2 = 0,\\
&2a_1k^2\omega^2 + a_0a_1k^2 + a_1k^2 - a_1\omega^2 = 0,\\
&-8a_2k^2\omega^2 - \frac{1}{2}a_1^2k^2 - a_0a_2k^2 - a_2k^2 + a_2\omega^2 = 0,\\
&2a_1k^2\omega^2 + a_1a_2k^2 = 0,\\
&-6a_2k^2\omega^2 - \frac{1}{2}a_2^2k^2 = 0
\end{aligned} \tag{4.9}$$

We find several different solutions and we take some of them:

$$\left\{\begin{array}{c}
a_0 \to -\frac{\sqrt{\frac{b_0^2\lambda^2\left(k\left(\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-k\lambda^2+4k\mu\right)+2\right)}{\lambda^2-4\mu}}}{\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}}, a_1 \to -\frac{(b_1\lambda+2b_0)\sqrt{\frac{b_0^2\lambda^2\left(k\left(\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-k\lambda^2+4k\mu\right)+2\right)}{\lambda^2-4\mu}}}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}},\\
a_2 \to -\frac{2b_1\sqrt{\frac{b_0^2\lambda^2\left(k\left(\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-k\lambda^2+4k\mu\right)+2\right)}{\lambda^2-4\mu}}}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}},\\
\omega \to -\frac{\sqrt{k^2\left(\lambda^2-4\mu\right)+2}}{\sqrt{\lambda^2-4\mu}}
\end{array}\right\},$$

$$\left\{\begin{array}{c}
a_0 \to \frac{\sqrt{\frac{b_0^2\lambda^2\left(k\left(\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-k\lambda^2+4k\mu\right)+2\right)}{\lambda^2-4\mu}}}{\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}}, a_1 \to \frac{(b_1\lambda+2b_0)\sqrt{\frac{b_0^2\lambda^2\left(k\left(\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-k\lambda^2+4k\mu\right)+2\right)}{\lambda^2-4\mu}}}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}},\\
a_2 \to \frac{2b_1\sqrt{\frac{b_0^2\lambda^2\left(k\left(\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-k\lambda^2+4k\mu\right)+2\right)}{\lambda^2-4\mu}}}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}},\\
\omega \to -\frac{\sqrt{k^2\left(\lambda^2-4\mu\right)+2}}{\sqrt{\lambda^2-4\mu}}
\end{array}\right\},$$

$$\left\{\begin{array}{c} a_0 \to -\frac{\sqrt{-\frac{b_0^2\lambda^2\left(k^2\left(\lambda^2-4\mu\right)+k\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-2\right)}{\lambda^2-4\mu}}}{\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}}, a_1 \to -\frac{(b_1\lambda+2b_0)\sqrt{-\frac{b_0^2\lambda^2\left(k^2\left(\lambda^2-4\mu\right)+k\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-2\right)}{\lambda^2-4\mu}}}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}}, \\ a_2 \to -\frac{2b_1\sqrt{-\frac{b_0^2\lambda^2\left(k^2\left(\lambda^2-4\mu\right)+k\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-2\right)}{\lambda^2-4\mu}}}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}}, \\ \omega \to \frac{\sqrt{k^2\left(\lambda^2-4\mu\right)+2}}{\sqrt{\lambda^2-4\mu}} \end{array}\right\},$$

$$\left\{\begin{array}{c} a_0 \to \frac{\sqrt{-\frac{b_0^2\lambda^2\left(k^2\left(\lambda^2-4\mu\right)+k\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-2\right)}{\lambda^2-4\mu}}}{\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}}, a_1 \to \frac{(b_1\lambda+2b_0)\sqrt{-\frac{b_0^2\lambda^2\left(k^2\left(\lambda^2-4\mu\right)+k\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-2\right)}{\lambda^2-4\mu}}}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}}, \\ a_2 \to \frac{2b_1\sqrt{-\frac{b_0^2\lambda^2\left(k^2\left(\lambda^2-4\mu\right)+k\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-2\right)}{\lambda^2-4\mu}}}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}}, \omega \to \frac{\sqrt{k^2\left(\lambda^2-4\mu\right)+2}}{\sqrt{\lambda^2-4\mu}} \end{array}\right\},$$

Case 1: If we take $\lambda^2 - 4\mu > 0, \mu \neq 0$

$$U_{(1,2)}(x,t) = \pm\frac{\sqrt{\frac{b_0^2\lambda^2\left(k\left(\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-k\lambda^2+4k\mu\right)+2\right)}{\lambda^2-4\mu}}\left(\lambda\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(C+kx-t\omega)\right)+\lambda^2-4\mu\right)}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(C+kx-t\omega)\right)+\lambda\right)} \tag{4.10}$$

$$V_{(1,2)}(x,t) = \frac{\sqrt{k^2\left(\lambda^2-4\mu\right)+2}\left(k\left(\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-k\lambda^2+4k\mu\right)+2\right)\left(\lambda\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(C+\xi)\right.\right.}{2\left(\lambda^2-4\mu\right)\left(\sqrt{k^2\left(\lambda^2-4\mu\right)+2}+k\sqrt{\lambda^2-4\mu}\right)\left(3k^2\left(\lambda^2-4\mu\right)-2\right)\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(C+\xi)\right.\right.} \tag{4.11}$$

$$U_{(3,4)}(x,t) = \pm\frac{\sqrt{-\frac{b_0^2\lambda^2\left(k^2\left(\lambda^2-4\mu\right)+k\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-2\right)}{\lambda^2-4\mu}}\left(\lambda\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(C+kx-t\omega)\right)+\lambda^2-4\mu\right)}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(C+kx-t\omega)\right)+\lambda\right)} \tag{4.12}$$

$$V_{(3,4)}(x,t) = -\frac{\sqrt{k^2\left(\lambda^2-4\mu\right)+2}\left(k^2\left(\lambda^2-4\mu\right)+k\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-2\right)\left(\lambda\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-}\right.\right.}{2\left(\lambda^2-4\mu\right)\left(\sqrt{k^2\left(\lambda^2-4\mu\right)+2}+k\sqrt{\lambda^2-4\mu}\right)\left(3k^2\left(\lambda^2-4\mu\right)-2\right)\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-}\right.\right.} \tag{4.13}$$

Case 2: If we take $\lambda^2 - 4\mu < 0, \mu \neq 0$

$$U_{(1,2)}(x,t)=\pm\frac{\sqrt{\frac{b_0^2\lambda^2\left(k\left(\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-k\lambda^2+4k\mu\right)+2\right)}{\lambda^2-4\mu}}\left(-\lambda\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(C+kx-t\omega)\right)+\lambda^2-4\mu\right)}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}\left(\lambda-\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(C+kx-t\omega)\right)\right)} \tag{4.14}$$

$$V_{(1,2)}(x,t)=\frac{\sqrt{k^2\left(\lambda^2-4\mu\right)+2}\left(k\left(\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-k\lambda^2+4k\mu\right)+2\right)\Big(-\lambda\sqrt{4\mu-\lambda^2}\tan\Big(\frac{1}{2}(C+\xi)}{2\left(\lambda^2-4\mu\right)\left(\sqrt{k^2\left(\lambda^2-4\mu\right)+2}+k\sqrt{\lambda^2-4\mu}\right)\left(3k^2\left(\lambda^2-4\mu\right)-2\right)\Big(\lambda-\sqrt{4\mu-\lambda^2}\tan\Big(\frac{1}{2}(C+} \tag{4.15}$$

$$U_{(3,4)}(x,t)=\pm\frac{\sqrt{-\frac{b_0^2\lambda^2\left(k^2\left(\lambda^2-4\mu\right)+k\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-2\right)}{\lambda^2-4\mu}}\left(-\lambda\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(C+kx-t\omega)\right)+\lambda^2-4\mu\right)}{b_0\lambda\sqrt{2-3k^2\left(\lambda^2-4\mu\right)}\left(\lambda-\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(C+kx-t\omega)\right)\right)} \tag{4.16}$$

$$V_{(3,4)}(x,t)-=\frac{\sqrt{k^2\left(\lambda^2-4\mu\right)+2}\left(k^2\left(\lambda^2-4\mu\right)+k\sqrt{\lambda^2-4\mu}\sqrt{k^2\left(\lambda^2-4\mu\right)+2}-2\right)\Big(-\lambda\sqrt{4\mu-\lambda^2}\tan\Big(\frac{1}{2}(C+\xi)}{2\left(\lambda^2-4\mu\right)\left(\sqrt{k^2\left(\lambda^2-4\mu\right)+2}+k\sqrt{\lambda^2-4\mu}\right)\left(3k^2\left(\lambda^2-4\mu\right)-2\right)\Big(\lambda-\sqrt{4\mu-\lambda^2}\tan\Big(\frac{1}{2}(C} \tag{4.17}$$

Case 3: If $\lambda^2-4\mu=0,\mu\neq 0,\lambda\neq 0$, we have no nontrivial solutions.

Case 4: If we take $\lambda^2-4\mu\neq 0,\mu=0,\lambda\neq 0$

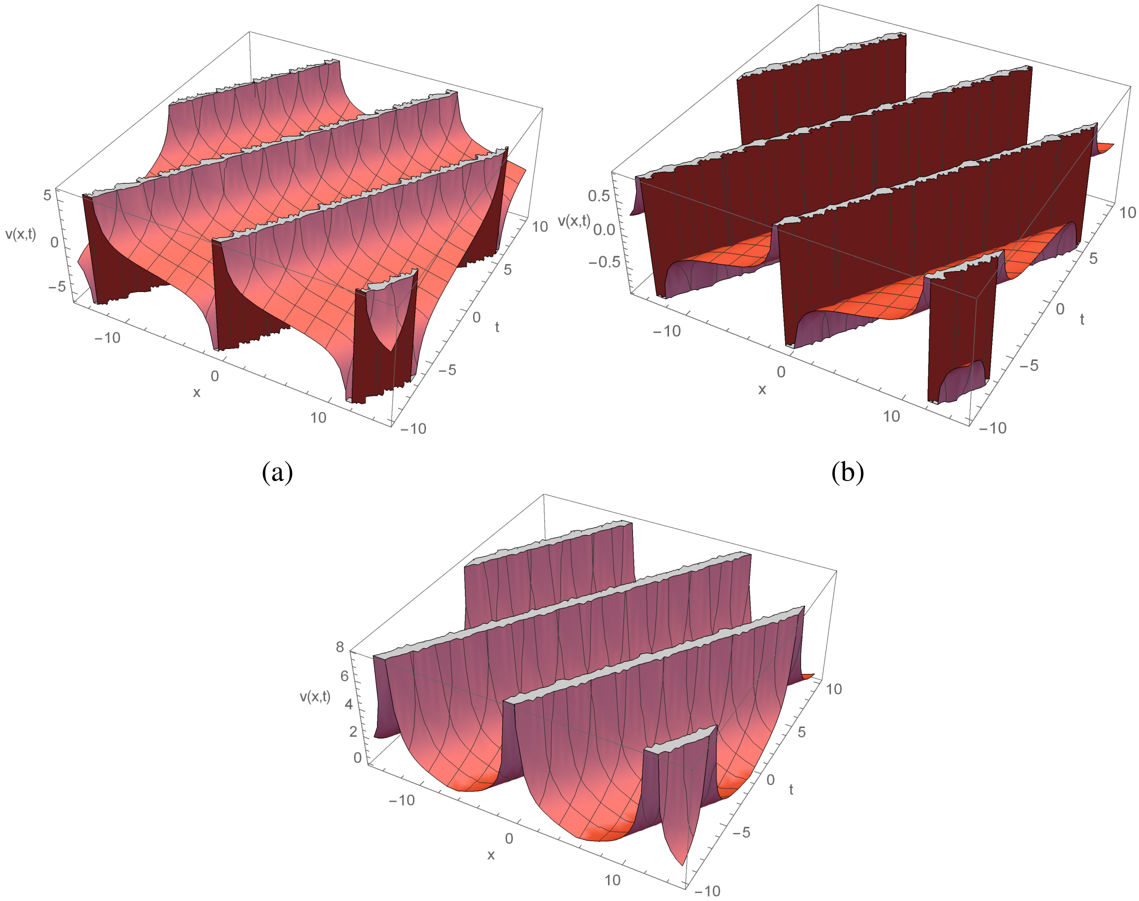
$$U_{(1,2)}(x,t)=\pm\frac{\sqrt{\frac{b_0^2\lambda^2\left(k\left(\sqrt{\lambda^2}\sqrt{k^2\lambda^2+2}-k\lambda^2\right)+2\right)}{\lambda^2}}\coth\left(\frac{1}{2}\lambda(C+kx-t\omega)\right)}{b_0\sqrt{2-3k^2\lambda^2}} \tag{4.18}$$

$$V_{(1,2)}(x,t)=\frac{\lambda^2\sqrt{k^2\lambda^2+2}\left(k\left(\sqrt{\lambda^2}\sqrt{k^2\lambda^2+2}-k\lambda^2\right)+2\right)\coth^2\left(\frac{1}{2}\lambda(C+kx-t\omega)\right)}{2\lambda^2\left(\sqrt{k^2\lambda^2+2}+k\sqrt{\lambda^2}\right)\left(3k^2\lambda^2-2\right)} \tag{4.19}$$

$$U_{(3,4)}(x,t)=\pm\frac{\sqrt{-\frac{b_0^2\lambda^2\left(k^2\lambda^2+k\sqrt{\lambda^2}\sqrt{k^2\lambda^2+2}-2\right)}{\lambda^2}}\coth\left(\frac{1}{2}\lambda(C+kx-t\omega)\right)}{b_0\sqrt{2-3k^2\lambda^2}} \tag{4.20}$$

$$V_{(3,4)}(x,t) = -\frac{\lambda^2\sqrt{k^2\lambda^2+2}\left(k^2\lambda^2 + k\sqrt{\lambda^2}\sqrt{k^2\lambda^2+2} - 2\right)\coth^2\left(\frac{1}{2}\lambda(C+kx-t\omega)\right)}{2\lambda^2\left(\sqrt{k^2\lambda^2+2} + k\sqrt{\lambda^2}\right)(3k^2\lambda^2-2)} \tag{4.21}$$

Case 5: If we take $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$, we have no solutions.



(a)

(b)

(c)

Figure 4.1: 3D Plot (a), (b) and (c) of some exact solutions of (4.1) given by(4.11),(4.13) and (4.15).

Chapter 5

Application of Extended $\left(\frac{G'}{G}\right)$-Expansion Method to the (1 + 1)-dimensional integro-differential Ito equation

The given equation is The (1 + 1)-dimensional integro-differential Ito equation [2, 6, 7, 8, 9, 45] , written as:

$$\frac{\partial u(x,t)}{\partial t^2} + \frac{\partial^4 u(x,t)}{\partial x^3 \partial t} + 6\frac{\partial u(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial t} + 3u(x,t)\frac{\partial^2 u(x,t)}{\partial x \partial t} + 3\frac{\partial^2 u(x,t)}{\partial x^2}\partial_x^{-1}\left(\frac{\partial u(x,t)}{\partial t}\right) = 0. \quad (5.1)$$

Where $\partial_x^{-1} := \int (.)\, dx$.

This is a nonlinear partial differntial equation (PDE) containing high-order integro-differential terms.

It is an extension of the classical Ito equation , which appears in contexts such as fluid dynamics and mathematical physics.

We consider the following transformation

$$\xi = kx - t\omega, \quad (5.2)$$

$$u(x,t) = U(\xi) = U(kx - t\omega), \quad (5.3)$$

$$v(x,t) = V(\xi) = V(kx - t\omega). \quad (5.4)$$

Where k and ω are a constants Let

$$v(x,t) = \int \frac{\partial u(x,t)}{\partial t}\, dx \quad (5.5)$$

So

$$U(\xi) = -\frac{k}{\omega} V(\xi) \tag{5.6}$$

Then we Using transformation Eq (5.5) , then transformed into the following ordinary differential equation

$$k^4 V''(\xi) - \frac{3k^3 V(\xi)^3}{\omega} - k\omega V(\xi) = 0 \tag{5.7}$$

Now we balancing V with V^3 yields $3N = N + 2$ so $N = 1$. Applying Extended $\left(\frac{G'}{G}\right)$-Expansion Method, the general solution takes the form

$$V(\xi) = a_{-1}(\frac{G'}{G}) + a_1(\frac{G}{G'}) + a_0 \tag{5.8}$$

by collecting all terms of the same order together, the lefthand side of equation is converted into polynomial and by settion each coefficient of each term to zero, we derive a system of algbrraic equation for $a_0, a_1, a_{-1}, k, \omega$

$$\begin{aligned}
& a_1 \lambda k^4 \mu + a_{-1} \lambda k^4 - \frac{3a_0^3 k^3}{\omega} - \frac{18 a_{-1} a_0 a_1 k^3}{\omega} - a_0 k \omega = 0, \\
& + 2a_{-1} k^4 \mu^2 - \frac{3a_{-1}^3 k^3}{\omega} = 0, \\
& + 3a_{-1} k^4 \lambda \mu - \frac{9a_{-1}^2 a_0 k^3}{\omega} = 0, \\
& a_{-1} + k^4 \lambda^2 + 2a_{-1} k^4 \mu - \frac{9a_{-1} a_0^2 k^3}{\omega} - \frac{9a_{-1}^2 a_1 k^3}{\omega} - a_{-1} k \omega = 0, \\
& a_1 + k^4 \lambda^2 + 2a_1 k^4 \mu - \frac{9a_{-1} a_1^2 k^3}{\omega} - \frac{9a_0^2 a_1 k^3}{\omega} - a_1 k \omega = 0, \\
& + 3a_1 k^4 \lambda - \frac{9a_0 a_1^2 k^3}{\omega} = 0, \\
& + 2a_1 k^4 - \frac{3a_1^3 k^3}{\omega} = 0
\end{aligned} \tag{5.9}$$

The resulting algebraic system Eq (5.9)is solved with the help of **Mathematica** to determine the values of the unkown constants $a_0, a_1, a_{-1}, k, \omega$

$$\left\{ a_0 \to -\frac{i k^2 \lambda \sqrt{\lambda^2 - 4\mu}}{2\sqrt{3}}, a_1 \to 0, a_{-1} \to -\frac{i k^2 \mu \sqrt{\lambda^2 - 4\mu}}{\sqrt{3}}, \omega \to -\frac{1}{2} k^3 \left(\lambda^2 - 4\mu\right) \right\}.$$

$$\left\{a_0 \to -\frac{ik^2\lambda\sqrt{\lambda^2-4\mu}}{2\sqrt{3}}, a_1 \to -\frac{ik^2\sqrt{\lambda^2-4\mu}}{\sqrt{3}}, a_{-1} \to 0, \omega \to -\frac{1}{2}k^3\left(\lambda^2-4\mu\right)\right\}.$$

$$\left\{a_0 \to \frac{ik^2\lambda\sqrt{\lambda^2-4\mu}}{2\sqrt{3}}, a_1 \to 0, a_{-1} \to \frac{ik^2\mu\sqrt{\lambda^2-4\mu}}{\sqrt{3}}, \omega \to -\frac{1}{2}k^3\left(\lambda^2-4\mu\right)\right\}.$$

$$\left\{a_0 \to \frac{ik^2\lambda\sqrt{\lambda^2-4\mu}}{2\sqrt{3}}, a_1 \to \frac{ik^2\sqrt{\lambda^2-4\mu}}{\sqrt{3}}, a_{-1} \to 0, \omega \to -\frac{1}{2}k^3\left(\lambda^2-4\mu\right)\right\}.$$

Now, by subsituting the values a_0, a_1, a_{-1}, ω and using Eq (5.8) in Eq (5.6), we obtain types of progressive wave solutions for Eq (5.1) as follows:

Case 1: If $\lambda^2 - 4\mu > 0$

Then the solutions are in the form

$$u_{(1,2)}(x,t) = \pm\frac{i\left(\frac{4\mu}{\frac{\sqrt{\lambda^2-4\mu}\left(A_2\coth\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_1\right)}{A_1\coth\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2}-\lambda}+\lambda\right)}{\sqrt{3}\sqrt{\lambda^2-4\mu}} \tag{5.10}$$

$$u_{(3,4)}(x,t) = \pm\frac{i\left(A_1\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{1}{2}k^3t\left(\lambda^2-4\mu\right)+kx\right)\right)+A_2\right)}{\sqrt{3}\left(A_2\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{1}{2}k^3t\left(\lambda^2-4\mu\right)+kx\right)\right)+A_1\right)} \tag{5.11}$$

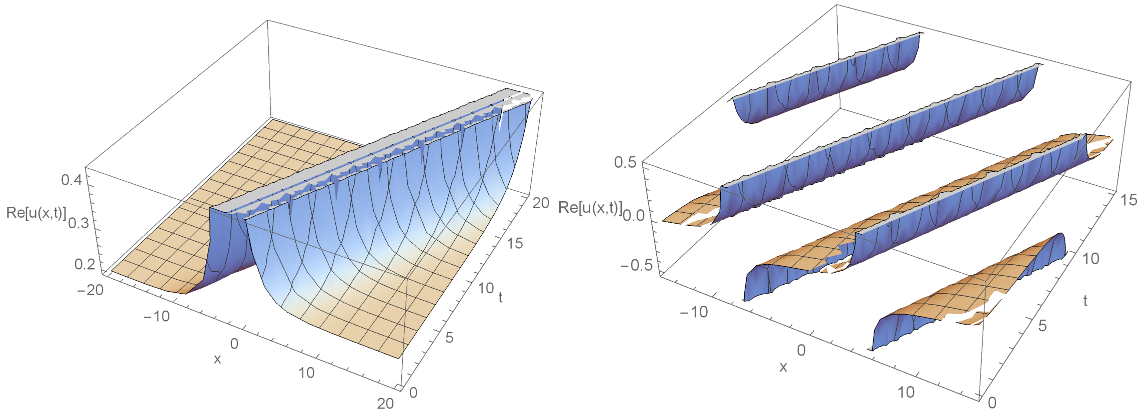
Case 2. if $\lambda^2 - 4\mu < 0$,

Then the solutions are in the form

$$u_{(1,2)}(x,t) = \pm\frac{ik^3\sqrt{\lambda^2-4\mu}\left(\lambda-\frac{4\mu}{\frac{\sqrt{4\mu-\lambda^2}\left(A_2\cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{1}{2}k^3t\left(\lambda^2-4\mu\right)+kx\right)\right)+A_1\right)}{A_1\cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{1}{2}k^3t\left(\lambda^2-4\mu\right)+kx\right)\right)-A_2}+\lambda}\right)}{2\sqrt{3}\omega} \tag{5.12}$$

$$u_{(3,4)}(x,t) = \pm\frac{\left\{\begin{matrix} ik^3\sqrt{-\left(\lambda^2-4\mu\right)^2}\left(A_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{1}{2}k^3t\left(\lambda^2-4\mu\right)+kx\right)\right)\right. \\ \left.+A_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{1}{2}k^3t\left(\lambda^2-4\mu\right)+kx\right)\right)\right) \end{matrix}\right\}}{\left\{\begin{matrix} 2\sqrt{3}\omega\left(A_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{1}{2}k^3t\left(\lambda^2-4\mu\right)+kx\right)\right)\right. \\ \left.-A_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{1}{2}k^3t\left(\lambda^2-4\mu\right)+kx\right)\right)\right) \end{matrix}\right\}} \tag{5.13}$$

Cases 3. if $\lambda^2 - 4\mu = 0$, there is no solutions



(a) (b)

Figure 5.1: 3D Plot (a), (b) of some exact solutions of (5.1) given by (5.10) and (5.12).

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