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MEMOIRE

In view of obtaining the MASTER's degree in:

Differential geometry

Title

EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR nth-ORDER THREE-POINT NONHOMOGENEOUS BOUNDARY VALUE PROBLEM

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Dedication

I dedicate this memory to my parents, in appreciation of their unconditional love and ongoing support, and for the sacrifices they made to provide the best conditions for my success. I also extend my greetings to my brothers and sisters, for their encouragement and reassuring presence by my side. A special thanks to my supervisor, Mr. NACERI MOSTEPHA, and to all my teachers in the Mathematics department. And to my friends, for their kindness, listening, and valuable advice throughout this journey. Also, to everyone who contributed, including the esteemed **Mr Medjahed Djilali** and the kind **Mr Guenedouz chikh**, for their efforts and the knowledge and guidance they provided. whether directly or indirectly, to the achievement of This wok.

Introduction

Ordinary differential equations are considered an essential tool in mathematical modeling, used to describe a wide range of phenomena in natural and engineering sciences, such as physics, biology, fluid mechanics, heat transfer, and others. These equations reflect the nature of the evolution of complex systems over time or across spatial variables by linking changes in physical quantities to their rates of change.

Among the various types of differential equations, higher-order boundary value problems stand out as a distinctive class, due to their ability to represent flexible and complex systems, especially when they involve unconventional conditions such as three or more points, or when they contain inhomogeneity in their structure, adding an additional analytical challenge in finding precise and interpretable solutions. This memo addresses the study of a specific problem, namely the 'existence and non-existence of positive solutions to the inhomogeneous third-order boundary value problem with three points.'

The study will also be expanded in Chapter Three to include the analysis of a fifth-order problem, reflecting the comprehensiveness of the analytical approach and its preparedness to deal with higher levels of complexity.

The importance of this topic is reflected in several aspects : From a theoretical perspective, the study focuses on the interaction between nonlinear differential equations, function spaces, and integrative effects, and how to construct solutions under complex conditions. As for the practical side, the results can be applied to a wide range of physical and engineering models, such as the behavior of flexible beams or thermal systems with variable sources. Regarding the analysis, the importance of tools such as fixed point theory and Green's function in addressing nonlinear and complex issues is emphasized.

The memory is divided into three main chapters, with each chapter relying on the preceding one in a logical sequence :

Chapter One : Fixed Point Theory This chapter is considered the theoretical foundation upon which the memo relies in addressing nonlinear issues. We review the most important fixed-point theorems, which are effectively used to prove the existence of solutions to differential equations, especially when transformed into equivalent integral equations. Among the prominent theories that we discuss : - Banach's theory relating to contraction mappings in complete spaces. - The Krasnoselskii theory, which combines contracting and compact operators in cones. - Schauder's theory, which applies to continuous and compact operators in normed spaces. We will discuss each of these theories in detail, highlighting the conditions for their realization, and analyzing how they can be used later in studying the existence of solutions.

Chapter Two : Green's Function This chapter is dedicated to the study of the Green's function associated with a three-point boundary problem, as it is considered a fundamental tool for transforming the differential equation into an integral equation suitable for analysis using fixed point theories. In this context, the following will be : - Deriving the appropriate Green's function form for the conditions of the studied problem. - Analyze the characteristics of this function in terms of continuity, sign, and integrability. - Using the Green's function to construct the operator associated with the solution, paving the way for applying the fixed point theory to this operator.

This chapter represents the link between abstract theory and practical application on a specific type of boundary problems.

Chapter three : to solve a fifth-degree problem

In this chapter, we continue the analysis by studying a fifth-order differential problem, which necessitates modifying the Green's function structure and flexibility in applying fixed point theories. This chapter aims to achieve several objectives : 1. Testing the ability of the analytical method used in the previous chapters to generalize to higher ranks. 2. Highlighting the mathematical differences between third-degree and fifth-degree analysis. 3. Provide practical or numerical cases when needed.

This chapter is considered an advanced step that demonstrates the effectiveness of the analytical approach and enhances its value in dealing with more complex models.

Chapitre 1

Reminder of the basics

This chapter is a series of results that will be useful for the rest of this thesis. We recall the main definitions and properties used in the remainder of the work, this to allow a more rapid assimilation of these notions. These consist essentially of functional analysis results. We thus find some properties for applications completely continuous in a Banach space, some properties of compact and relatively compact operators in spaces continuous applications. Â Also given various fixed-point theorems, including, Schauder theorem, the fixed point theorem of Guo-Krasnosel'skii recently, Leggett-Williams fixed point notions that we will Â find all along this work. It also recalls the definitions of the upper solution and the lower solution concepts that will be introduced and used in Chapter II and Chapter VI .

1.1 Contraction Mapping

Assume that B is a closed subset of a Banach space X and $T : B \to B$ is a contraction mapping, i.e., there exists $\theta \in [0, 1)$ such that

$$\|Tx - Ty\| \le \theta \|x - y\| \quad \forall x, y \in B.$$

Then there is a unique $x \in B$ such that $x = Tx$.

1.2 The Brouwer Fixed Point Theorem

Any continuous map from the two dimensional unit disc to itself must have (at least) one fixed point.

1.2.1 Example

Any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. Consider the function $g(x) = f(x) - x$, which is also continuous on $[0, 1]$. We have $g(0) = f(0) \in [0, 1]$, so $g(0) \ge 0$. Also, $g(1) = f(1) - 1 \in [-1, 0]$, so $g(1) \le 0$. Thus, we have $g(1) \leq 0 \leq g(0)$. By the I.V.T., there is some $c \in [0,1]$ such that $g(c) = 0$. This means $f(c) = c$, so c is a fixed-point of $f(x)$.

1.3 The Schauder's Fixed Point Theorem

We recall that a set is precompact if every sequence in the set contains a convergent subsequence. A set is compact if it is precompact and closed.

Let X be a Banach space, B be a closed and convex subset of X, and $T : B \mapsto B$ be a continuous map. Then T has at least one fixed point in B if the range $T(B) := \{Tx; x \in B\}$ is precompact.

Note that if B is compact, then automatically $T(B) \subset B$ is precompact.

In both the contraction and the Schauder fixed point theorems, the condition that T maps B into itself is usually the key difficulties in applications.

The central point of the Schauder's fixed point theorem is the compactness of the image TB. Quite often one takes a bounded B and shows that T is compact. In other cases, one simply takes a compact B (then one has to be very careful about the continuity of T).

1.4 The Leray–Schauder Fixed Point Theorem

Let X be a Banach space and $T : X \to X$ be compact and continuous. Suppose there exists a positive constant M such that

$$\|x\| \leq M \text{ whenever } \exists\, \sigma \in [0,1] \text{ such that } x = \sigma T x.$$

Then has a fixed point in X.

The Leray–Schauder's fixed point theorem is also known as the Schaefer's fixed point theorem. Its advantage over the Schauder's fixed point theorem for applications is that we do not have to identify a convex closed subset which T maps into itself.

1.5 Arzela-Ascoli

Let $K \subset R^n$ be a compact set. A subset $F \in C(K)$ is relatively compact if and only if it is pointwise bounded and equicontinuous, where $C(K)$ denotes the space of all continuous functions on K.

1.6 Arzela-Ascoli

If a sequence $\{fm\}_1^\infty$ in $C(K)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.

1.7 The Krasnosel'skii Fixed Point Theorem

Let M be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose A and B map M into S such that

(i) $A\Phi + B\Psi \in M$ for all $\Phi, \Psi \in M$;

(ii) A is continuous and AM is contained in a compact set ;
(iii) B is a contraction.
Then there exists a $\Phi \in M$ with $\Phi = A\Phi + B\Phi$.

1.8 The Schaefer Fixe Poind Theorem

Let $(S, \|\cdot\|)$ be a normed space, and let H a continuous mapping from S into S which is compact on each bounded subset X of S. Then either
(i) the equation $x = \lambda Hx$ has a solution for $\lambda = 1$, or
(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

1.9 The Krasnosel'skii-Schaefer Fixed Poind Theorem

Let $(S, \|\cdot\|)$ be a Banach space. Suppose $B : S \to S$ is a contraction map, and $A : S \to S$ is continuous and maps bounded sets into compact sets. Then either
(i) $x = \lambda B(\frac{x}{\lambda}) + \lambda Ax$ has a solution in S for $\lambda = 1$, or
(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

1.9.1 Definition

Let E be a real Banach space.A nonempty closed convex set $K \subset E$ is called cone if it satisfies the following two conditions :

1. $(\alpha u + \beta v) \in K$ for all $u, v \in K$ and all $\alpha, \beta \geq 0$
   and
2. $u \in K$ and $-u \in K \Longrightarrow u \equiv 0$

1.10 The Guo-Krasnosel'skii Fixed Point Theorem

Let $(E, \|.\|)$ be a Banach space and $K \subset E$ be cone in E. Assume that Ω_1 and Ω_2 are open with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$,and let
$T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ be a completely continuous operator such that either :

i) $\|Tu\| \leq \|u\|$,$\forall u \in K \cap \delta\Omega_1$ and $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \delta\Omega_2$

ii) $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \delta\Omega_1$ and $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \delta\Omega_2$,

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$

1.10.1 Definition

Given a cone K in real Banach space E,a map α is said to be nonnegative continuous concave(resp.convex)functional on K provided that :

1) $\alpha : K \to [0, \infty)$ is continuous

2) $\alpha(tx+(1-t)y) \geq t\alpha x+(1-t)\alpha y[resp. \quad \alpha(tx+(1-t)y) \leq t\alpha x+(1-t)\alpha y]$ For all $x, y \in K \quad and \quad t \in [0,1]$

1.10.2 Definition

For all $x, y \in K$ and $t \in [0,1]$. Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on K.Define the convex sets P_r and $P(\alpha, a, b)$ by
$P_r = \{x \in K/\|x\| < r\}$,$P(\alpha, a, b) = \{x \in K/a \leq \alpha(x); \|x\| \leq b\}$

1.11 The Leggett-Williams Fixed Point Theorem

Let $T : \overline{P_c} \longrightarrow \overline{P_c}$ be a completely continuous operator and let α be a nonnegative continuous concave functional on K such that $\alpha(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < a < b < d \leq c$ such that

1) $\{x \in P(\alpha, b, d)/\alpha(x) > b\} \neq \emptyset\emptyset$ and $\alpha(Tx) > b$ for $x \in P(\alpha, b, d)$

2) $\|Tx\| < a$ for $\|x\| \leq a$.

3) $\alpha(Tx) > b$ for $x \in P(\alpha, b, c)$ with $\|Tx\| > d$.

Then T has at least three fixed points x_1, x_2, x_3 such that $\|x_1\| < a$,$b < \alpha(x_2)$ and $\|x_3\| > a$ with $\alpha(x_3) < b$

1.12 Topological degree of Brouwer

Now, we give the construction of Brouwer degree in this section as follows :

1.12.1 Definition

Let $\Omega \subset R^N$ be open and bounded and $f \in C^1(\overline{\Omega})$. if $p \notin f(\partial\Omega)$ and $J_f(p) \neq 0$, then we define

$$deg(f, \Omega, p) = \Sigma_{x \in f^{-1}(p)} sgn J_f(x),$$

where $deg(f, \Omega, p) = 0$ if $f^{-1}(p) = \emptyset$

1.12.2 Theorem

Let $\Omega \subset R^N$ be an open bounded subset and $f : \overline{\Omega} \to R^N$ be a continuous mapping. if $p \notin f(\partial\Omega)$, then there exists an integer $deg(f, \Omega, p)$ satisfying the following properties :

1. (Normality) $deg(I, \Omega, p) = 1$ if and only if $p \in \Omega$, where I denotes the identity mapping ;

2. (Solvability) If $deg(I, \Omega, p) \neq 0$, then $f(x) = p$ has a solution in Ω;
3. (Homotopy) If $f_t(x) : [0,1] \times \overline{\Omega} \to R^N$ is continuous and $p \notin \cup_{t \in [0,1]} f_t(\partial\Omega)$, then $deg(f_t, \Omega, p)$ does not depend on $t \in [0,1]$;
4. (Additivity) Suppose that Ω_1, Ω_2 are two disjoint open subsets of Ω and $p \notin f(\overline{\Omega} - \Omega_1 \cup \Omega_2)$. Then $deg(f, \Omega, p) = deg(f, \Omega_1, p) + deg(f, \Omega_2, p)$;

1.12.3 Theorem

Let $C \subset R^N$ be a nonempty bounded closed convex subset and $f : C \to C$ be a continuous mapping. Then f has a fixed point in C.

1.12.4 Theorem

Let $f : R^n \to R^n$ be continuous mapping and $0 \in \Omega \subset R^n$ with Ω an open bounded subset. if $(f(x), x) > 0$ for all $x \in \partial\Omega$, then $deg(f, \Omega, 0) = 1$.

1.13 Topological degree of Leray Schauder

In this section, we construct the Leray Schauder degree. First, we need the following result on the approximation of a compact mapping by finite dimensional mappings.

1.13.1 Definition

Let E be a real Banach space, $\Omega \subset E$ be an open bounded subset and $T : \overline{\Omega} \to E$ be a continuous compact mapping. Then, for any $\epsilon > 0$, there exist a finite dimensional space F and a continuous mapping $T_\epsilon : \overline{\Omega} \to E$ such that

$$\|T_\epsilon x - T_\epsilon x\| < \epsilon forall x \in T_\epsilon$$

1.13.2 Theorem

The Leray Schauder degree has the following properties

1. (Normality) $deg(I, \Omega, 0) = 1$ if and only if $0 \in \Omega$;

2. (Solvability) If $deg(I - T, \Omega, 0) \neq 0$, then $Tx = x$ has a solution in Ω;

3. (Homotopy) Let $T_t : [0,1] \times \overline{\Omega} \to E$ be continuous compact and $T_t x \neq x$ for all $(t, x) \in [0,1] \times \overline{\Omega}$. Then $deg(I - T_t, \Omega, 0)$ does not depend on $t \in [0,1]$;

4. (Additivity) Let Ω_1, Ω_2 be two disjoint open subsets of Ω and $0 \notin (I - T)(\overline{\Omega} - \Omega_1 \cup \Omega_2)$. Then $deg(I - T, \Omega, 0) = deg(I - T, \Omega_1, 0) + deg(I - T, \Omega_2, 0)$.

1.13.3 Theorem

Let $C \subset E$ be a nonempty bounded closed convex subset and $T : C \to C$ be a continuous mapping. Then T has a fixed point in C.

Chapitre 2

Green function

1. Introduction

Let $\Omega \subseteq R^n$ be a domain, and let L be a **linear differential operator**. A function $G(x, \xi)$ is called a *Green's function* for the operator L with respect to some boundary conditions if it satisfies :

$$L_x G(x, \xi) = \delta(x - \xi)$$

The Greenâs function allows us to construct the solution to an inhomogeneous linear differential equation :

$$Lu(x) = f(x)$$

using :

$$u(x) = \int_\Omega G(x, \xi) f(\xi)\, d\xi$$

So, $G(x, \xi)$ plays the role of a *kernel* that describes how the system responds to a point source.

Examples of L :
[label=–](L = d$^2dx^2$ â ordinary differential equations. $L = \Delta = \nabla^2$ â Laplacian in partial differential equations.

2. Properties of Green function

For all $(t, s) \in [0, 1] \times [0, 1]$, we have :

$$\frac{1}{4!} t^4 (1 - s) \leq G(t, s) \leq s(1 - s)$$

Proof. It is obvious that $G(t, s) \geq 0$ and $G(t, s) \leq G(1, s)$.

If $0 \leq t \leq s \leq 1$, then

$$G(t, s) = \frac{1}{4!} t(1 - s) \leq \frac{1}{4!} s(1 - s)$$

and

$$G(t,s) = \frac{1}{4!}t(1-s) \geq \frac{1}{4!}t^4(1-s)$$

If $0 \leq s \leq t \leq 1$, then

$$\frac{G(t,s)}{G(1,s)} = \frac{t^4(1-s)-(t-s)^4}{(1-s)-(1-s)^4} = \frac{t^4(1-s)-(t-s)^4}{(1-s)(1-(1-s)^3)} \geq t^4$$

so

$$G(t,s) \geq t^4 G(1,s) = \frac{t^4}{4!}[1-(1-s)^4] \geq \frac{t^4}{4!}s$$

and

$$G(t,s) = \frac{t^4(1-s)-(t-s)^4}{4!} = \frac{1}{4!}\left[t^4(1-s)+2t^2s(1-s)-s^4(1-s)\right] \leq s(1-s)$$

3.Examples

either $y \in [a,b]$. We consider the following boundary problem :

$$\begin{cases} u^{(3)}(t)+y(t)=0 \ , \ t \in J=[a,b] \\ u(a)=0, u^{'}(a)=0, u(b)=0 \end{cases} \tag{2.1}$$

The problem (2.2) is equivalent to the integral equation :

$$u(t) \quad = \quad \int_a^b H(t,s)y(s)ds \tag{2.2}$$

or H the Green's function associated with the problem (2.1), defiin ned by définie de $[a,b]\times[a,b]$ in $[0,\infty)$

and given by the following expression :

$$H(t,s) = \begin{cases} \frac{(b-s)^2(t-a)^2-(b-a)^2(t-s)^2}{2(b-a)^2} & \text{si } a \leq s \leq t \leq b \\ \\ \frac{(b-s)^2(t-a)^2}{2(b-a)^2} & \text{si } a \leq t \leq s \leq b \end{cases} \tag{2.3}$$

Proof. . We give the formal expression of H(t,s) by

$$H(t,s) = \begin{cases} \alpha_0 t^2+\alpha_1 t+\alpha_2 & \text{si } a \leq s \leq t \leq b \\ \\ \beta_0 t^2+\beta_1 t+\beta_2 & \text{si } a \leq t \leq s \leq b \end{cases}$$

we place

$$c_i \quad = \quad \beta_i - \alpha_i$$

According to the properties of the Green's function, we obtain the following system :

$$\begin{cases} c_0 s^2 + c_1 s + c_2 = 0 \\ 2c_0 s + c_1 = 0 \\ 2c_0 = 1 \end{cases}$$

With a simple calculation, we find

$$c_0 = \frac{1}{2}, c_1 = -s, c_2 = \frac{s^2}{2} \tag{2.4}$$

By using the boundary conditions, we obtain the system.

$$\begin{cases} \beta_0 a^2 + \beta_1 a + \beta_2 = 0 \\ 2\beta_0 a + \beta_1 = 0 \\ \alpha_0 b^2 + \alpha_1 b + \alpha_2 = 0 \end{cases}$$

Using (2.5), we obtain the following Cramer's system :

$$\begin{cases} \beta_0 a^2 + \beta_1 a + \beta_2 = 0 \\ \beta_0 b^2 + \beta_1 b + \beta_2 = d \\ 2\beta_0 a + \beta_1 = 0 \end{cases}$$

either

$$\mathcal{A} = \begin{pmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ 2a & 1 & 0 \end{pmatrix} \quad , \quad \mathcal{B} = \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}$$

where

$$d = \frac{b^2}{2} - bs + \frac{s^2}{2} \quad , \quad detA = (b-a)^2 \neq 0$$

We obtain :

$$\beta_0 = \frac{d}{detA} \quad , \quad \alpha_0 = \frac{d}{detA} - \frac{1}{2}$$

$$\beta_1 = \frac{-2ad}{detA} \quad , \quad \alpha_1 = \frac{-2ad}{detA} + s$$

$$\beta_2 = \frac{a^2 d}{detA} \quad , \quad \alpha_2 = \frac{a^2 d}{detA} - \frac{s^2}{2}$$

We substitute in the expression of H(t,s) we obtain :

I. For $t \leq s$

$$\begin{aligned} H(t,s) &= \frac{d}{detA}t^2 - \frac{2ad}{detA}t + \frac{a^2d}{detA} \\ H(t,s) &= \frac{d}{detA}(t^2 - 2at + a^2) \\ H(t,s) &= \frac{(b^2 - 2bs + s^2)(t^2 - 2at + a^2)}{2detA} \end{aligned}$$

thus

$$H(t,s) = \frac{(b-s)^2(t-a)^2}{2(b-a)^2} \quad ; \quad a \leq t \leq s \leq b$$

II. For $s \leq t$

$$\begin{aligned} H(t,s) &= (\frac{d}{detA} - \frac{1}{2})t^2 + (\frac{-2ad}{detA} + s)t + (\frac{a^2d}{detA} - \frac{s^2}{2}) \\ H(t,s) &= \frac{d}{detA}(t^2 - 2at + a^2) - \frac{t^2}{2} + st - \frac{s^2}{2} \\ H(t,s) &= \frac{(b^2 - 2bs + s^2)(t^2 - 2at + a^2) - detA(t^2 - 2st + s^2)}{2detA} \end{aligned}$$

thus

$$H(t,s) = \frac{(b-s)^2(t-a)^2 - (b-a)^2(t-s)^2}{2(b-a)^2} \quad ; \quad a \leq s \leq t \leq b$$

2.0.1 Lemma

H(t,s) has the following properties :

- $H(t,s) \leq k(s)\ \forall t, s \in [a,b]\ avec\ k(s) = \frac{(b-s)^2(s-a)}{(b-a)}$ et $\max_{a \leq s \leq b} |k(s)| = \frac{4}{27}(b-a)^2$
- $H_t^{'}(t,s) \leq l(s)\ \forall t, s \in [a,b]\ avec\ l(s) = \frac{(b-s)^2(b-a)+(b-a)^2(b-s)}{(b-a)^2}$ et $\max_{a \leq s \leq b} |l(s)| = \frac{b-a}{4}$
- $H_{tt}^{''}(t,s) \leq p(s)\ \forall t, s \in [a,b]\ avec\ p(s) = \frac{(b-s)^2}{(b-a)^2}$ et $\max_{a \leq s \leq b} |h(s)| = 1$

Proof of the lemma.

$$H(t,s) = \begin{cases} \frac{[(b-s)(t-a)+(b-a)(t-s)][(b-t)(s-a)]}{2(b-a)^2} & \text{si } a \leq s \leq t \leq b \\ \\ \frac{(b-s)^2(t-a)^2}{2(b-a)^2} & \text{si } a \leq t \leq s \leq b \end{cases} \tag{2.6}$$

$$H(t,s) \leq \begin{cases} \frac{2(b-s)^2(b-a)(s-a)}{2(b-a)^2} & \text{si } a \leq s \leq t \leq b \\ \\ \frac{(b-s)^2(s-a)^2}{2(b-a)^2} & \text{si } a \leq t \leq s \leq b \end{cases}$$

$$\leq \frac{(b-s)^2(s-a)}{(b-a)} \qquad \forall t,s \in [a,b]$$

Let $k(s) = \frac{(b-s)^2(s-a)}{(b-a)}$.The function $k(s)$ admits a maximum at the point $s = \frac{b+2a}{3}$ with

$\max_{a \leq s \leq b} |k(s)| = \frac{4}{27}(b-a)^2$

$$H_t{}'(t,s) = \begin{cases} \frac{(b-s)^2(t-a)-(b-a)^2(t-s)}{(b-a)^2} & \text{si } a \leq s \leq t \leq b \\ \\ \frac{(b-s)^2(t-a)}{(b-a)^2} & \text{si } a \leq t \leq s \leq b \end{cases}$$

$$H_t{}'(t,s) \leq \begin{cases} \frac{(b-s)^2(t-a)+(b-a)^2(t-s)}{(b-a)^2} & \text{si } a \leq s \leq t \leq b \\ \\ \frac{(b-s)^2(t-a)}{(b-a)^2} & \text{si } a \leq t \leq s \leq b \end{cases}$$

$$H_t{}'(t,s) \leq \begin{cases} \frac{(b-s)^2(b-a)+(b-a)^2(b-s)}{(b-a)^2} & \text{si } a \leq s \leq t \leq b \\ \\ \frac{(b-s)^2(s-a)}{(b-a)^2} & \text{si } a \leq t \leq s \leq b \end{cases}$$

$$\leq \frac{(b-s)^2(b-a)+(b-a)^2(b-s)}{(b-a)^2} \qquad \forall t,s \in [a,b]$$

either $l(s) = \frac{(b-s)^2(b-a)+(b-a)^2(b-s)}{(b-a)^2}$.the function $l(s)$ admits a maximum at the point $s = \frac{3b-a}{2}$ with
$\max_{a \leq s \leq b} |l(s)| = \frac{b-a}{4}$

$$H_{tt}{}''(t,s) = \begin{cases} \frac{(b-s)^2-(b-a)^2}{(b-a)^2} & \text{si } a \leq s \leq t \leq b \\ \\ \frac{(b-s)^2}{(b-a)^2} & \text{si } a \leq t \leq s \leq b \end{cases}$$

$$\leq \frac{(b-s)^2}{(b-a)^2} \quad \forall t, s \in [a,b]$$

either $p(s) = \frac{(b-s)^2}{(b-a)^2}$.the function $p(s)$ admits a maximum at the point $s = a$ with
$\max_{a \leq s \leq b} |p(s)| = 1$
We can thus provide the following estimates :

$$\begin{aligned}
\max_{a \leq t,s \leq b} |H(t,s)| &\leq \frac{4}{27}(b-a)^2 \\
\max_{a \leq t,s \leq b} |H_t^{'}(t,s)| &\leq \frac{1}{4}(b-a) \\
\max_{a \leq t,s \leq b} |H_{tt}^{''}(t,s) &\leq 1
\end{aligned}$$

According to the expression (2.7), it is observed that $H(t,s) \geq 0$ for all $t \in [a,b]$.

2.0.2 Lemma

H(t,s)$\geq \phi(t)k(s), \forall t, s \in [a,b]$.
Where

$$\phi(t) = \begin{cases} \frac{(t-a)^2}{2(b-a)^2} & \text{si} \quad a \leq t \leq \frac{a+b}{2} \\ \\ \frac{(b-t)(t-a)}{2(b-a)^2} & \text{si} \quad \frac{a+b}{2} \leq t \leq b \end{cases}$$

Proof. On one hand if $s = a$ or $s = b$ on a $H(t,s) = 0 \geq \phi(t)0$ so the result is trivial. On the other hand, we have

$$\frac{H(t,s)}{k(s)} = \begin{cases} \frac{(b-s)^2(t-a)^2-(b-a)^2(t-s)^2}{2(s-a)(b-s)^2(b-a)} & \text{si } a \leq s \leq t \leq b \\ \\ \frac{(t-a)^2}{2(s-a)(b-a)} & \text{si } a \leq t \leq s \leq b \end{cases}$$

$$\frac{H(t,s)}{k(s)} = \begin{cases} \frac{[(b-s)(t-a)+(b-a)(t-s)][(b-t)(s-a)]}{2(s-a)(b-s)^2(b-a)} & \text{si} \quad a \leq s \leq t \leq b \\ \\ \frac{(t-a)^2}{2(s-a)(b-a)} & \text{si} \quad a \leq t \leq s \leq b \end{cases}$$

$$\frac{H(t,s)}{k(s)} \geq \begin{cases} \frac{(b-s)(t-a)(b-t)(s-a)}{2(s-a)(b-s)^2(b-a)} & \text{si} \quad a \leq s \leq t \leq b \\ \\ \frac{(t-a)^2}{2(s-a)(b-a)} & \text{si} \quad a \leq t \leq s \leq b \end{cases}$$

$$\frac{H(t,s)}{k(s)} \geq \begin{cases} \frac{(t-a)(b-t)}{2(b-a)^2} & \text{si} \quad a \leq s \leq t \leq b \\ \\ \frac{(t-a)^2}{2(b-a)^2} & \text{si} \quad a \leq t \leq s \leq b \end{cases}$$

like

$$\text{for } t \in [a, \tfrac{a+b}{2}]$$

and

$$(t-a)^2 \geq (t-a)(b-t) \text{for } t \in [\tfrac{a+b}{2}, b]$$

We obtain :

$$\phi(t) = \begin{cases} \frac{(t-a)^2}{2(b-a)^2} & \text{si} \quad a \leq t \leq \frac{a+b}{2} \\ \\ \frac{(b-t)(t-a)}{2(b-a)^2} & \text{si} \quad \frac{a+b}{2} \leq t \leq b \end{cases}$$

2.1 Example 2

2.1.1 Lemma

1 Suppose that $p(t) \in C(0,1)$ and $p(t) > 0$. Then the linear boundary value problem

$$\begin{cases} u^{(5)}(t) + p(t) = 0, & 0 < t < 1, \\ \\ u(0) = u'(0) = u''(0) = u'''(0) = 0, \\ \\ u'''(1) - \alpha u'''(\eta) = \lambda. \end{cases} \tag{2.7}$$

has a unique solution, which can be expressed by

$$u(t) = \int_0^1 G(t,s)p(s)ds + \frac{\alpha t^4}{4!(1-\alpha\eta)} \int_0^1 K(\eta,s)p(s)ds + \frac{\lambda t^4}{4!(1-\alpha\eta)}$$

where

$$G(t,s)=\begin{cases}\frac{t^4(1-s)-(t-s)^4}{4!}, & 0\le s\le t\le 1,\\ \frac{t^4(1-s)}{4!}, & 0\le t\le s\le 1.\end{cases}$$

and

$$\frac{\partial}{\partial t}G(t,s)=\begin{cases}\frac{t^3(1-s)-(t-s)^3}{6}, & 0\le s\le t\le 1,\\ \frac{t^3(1-s)}{6}, & 0\le t\le s\le 1.\end{cases}$$

and

$$\frac{\partial^2}{\partial t^2}G(t,s)=\begin{cases}\frac{t^2(1-s)-(t-s)^2}{2}, & 0\le s\le t\le 1,\\ \frac{t^2(1-s)}{2}, & 0\le t\le s\le 1.\end{cases}$$

and

$$K(t,s)=\frac{\partial^3}{\partial t^3}G(t,s)=\begin{cases}s(1-t), & 0\le s\le t\le 1,\\ t(1-s), & 0\le t\le s\le 1.\end{cases}$$

Proof. In fact, if $u(t)$ is a solution of Problem (4.2) by the Taylor expansion formula, we have

$$u(t)=C_0+C_1t+\frac{C_2}{2!}t^2+\frac{C_3}{3!}t^3+\frac{C_4}{4!}t^4-\frac{1}{4!}\int_0^t(t-s)p(s)ds$$

which together with the boundary condition implies $C_0=C_1=C_2=C_3=0$

$$C_4=\frac{1}{1-\alpha\eta}\int_0^1(1-s)p(s)ds-\frac{\alpha}{1-\alpha\eta}\int_0^\eta(\eta-s)p(s)ds+\frac{\lambda}{1-\alpha\eta}$$

Therefore

$$\begin{aligned}u(t)&=-\frac{1}{4!}\int_0^t(t-s)^4p(s)ds+\frac{t^4}{4!(1-\alpha\eta)}\int_0^1(1-s)p(s)ds-\frac{\alpha t^4}{4!(1-\alpha\eta)}\int_0^\eta(\eta-s)p(s)ds\\&+\frac{\lambda t^4}{4!(1-\alpha\eta)}\\&=\frac{1}{4!}\int_0^t[t^4(1-s)-(t-s)^4]p(s)ds+\frac{t^4}{4!}\int_t^1(1-s)p(s)ds\\&+\frac{\alpha t^4}{4!(1-\alpha\eta)}[\int_0^\eta s(1-\eta)p(s)ds+\int_\eta^1\eta(1-s)p(s)ds]+\frac{\lambda t^4}{4!(1-\alpha\eta)}\\&=\int_0^1G(t,s)p(s)ds+\frac{\alpha t^4}{4!(1-\alpha\eta)}\int_0^1K(\eta,s)p(s)ds+\frac{\lambda t^4}{4!(1-\alpha\eta)}.\end{aligned}$$

The proof is complete.

2.1.2 Properties of Green's function

For all $(t,s) \in [0,1] \times [0,1]$, we have

$$\frac{1}{4!}t^4 s(1-s) \leq G(t,s) \leq s(1-s)$$

Proof. It is obvious that $G(t,s) \geq 0$ and $G(t,s) \leq G(1,s)$.

If $0 \leq t \leq s \leq 1$, then

$$G(t,s) = \frac{1}{4!}t^4(1-s) \leq \frac{1}{4!}s^4(1-s) \leq s(1-s)$$

and

$$G(t,s) = \frac{1}{4!}t^4(1-s) \geq \frac{1}{4!}t^4 s(1-s)$$

If $0 \leq s \leq t \leq 1$, then

$$\frac{G(t,s)}{G(1,s)} = \frac{t^4(1-s)-(t-s)^4}{(1-s)-(1-s)^4} \geq \frac{t^4(1-s)-(t-ts)^4}{(1-s)-(1-s)^4} \geq t^4$$

so

$$G(t,s) \geq t^4 G(1,s) = \frac{t^4}{4!}(1-s)[1-(1-s)^3] \geq \frac{t^4}{4!}s(1-s)$$

and

$$G(t,s) = \frac{t^4(1-s)-(t-s)^4}{4!} \leq \frac{1}{4!}s[t^3(1-s)+2t^2(1-s)] \leq s(1-s)$$

.

2.2 Example 3

If $y \in C[0,1]$,then the problem

$$\begin{cases} -w'' + k^2 w = y(t) \ , \ t \in [0,1] \\ w(0) = 0, w(1) = 0 \end{cases} \tag{2.8}$$

has a unique positive solution w and w can be expressed in the form

$$w(t) = \int_0^1 H_1(t,s)y(s)ds \tag{2.9}$$

where

$$H_1(t,s) = \begin{cases} \frac{\sinh(ks)\sinh(k(1-t))}{k\sinh(k)} \ , \ 0 \leq s \leq t \\ \frac{\sinh(kt)\sinh(k(1-s))}{k\sinh(k)} \ , \ t \leq s \leq 1 \end{cases} \tag{2.10}$$

$H_1(t,s)$ has the following properties

- $H_1(t,s) \geq 0, \forall t,s \in [0,1]$
- $\frac{k}{\sinh(k)}H_1(t,t)H_1(s,s) \leq H_1(t,s) \leq H_1(t,t)$

Proof. It is obvious that $H_1(t,s)$ is nonnegative.Moreover,than

$$H_1(t,s) = \begin{cases} \frac{\sinh(ks)\sinh(k(1-t))}{k\sinh(k)} \ , 0 \le s \le t \\ \\ \frac{\sinh(kt)\sinh(k(1-s))}{k\sinh(k)} \ , t \le s \le 1 \end{cases} \tag{2.11}$$

$$H_1(t,s) \le \frac{1}{k\sinh(k)} \begin{cases} \sinh(kt)\sinh(k(1-t)) \ , 0 \le s \le t \\ \\ \sinh(kt)\sinh(k(1-t)) \ , t \le s \le 1 \end{cases} \tag{2.12}$$

$$H_1(t,s) \le \frac{k\sinh(k)}{\sinh(kt)\sinh(k(1-t))} \qquad \forall \quad t,s \in [0,1]$$

$$H_1(t,s) \le H_1(t,t) \ \ \forall \ t \in [0,1]$$

Then

$$\frac{H_1(t,s)}{H_1(t,t)H_1(s,s)} = \begin{cases} \frac{k\sinh(k)}{\sinh(kt)\sinh(k(1-s))} \ , 0 \le s \le t \\ \\ \frac{k\sinh(k)}{\sinh(ks)\sinh(k(1-t))} \ , t \le s \le 1 \end{cases} \tag{2.13}$$

$$\frac{H_1(t,s)}{H_1(t,t)H_1(s,s)} \ge \begin{cases} \frac{k\sinh(k)}{\sinh(k)\sinh(k)} \ , 0 \le s \le t \\ \\ \frac{k\sinh(k)}{\sinh(k)\sinh(k)} \ , t \le s \le 1 \end{cases} \tag{2.14}$$

$$\frac{H_1(t,s)}{H_1(t,t)H_1(s,s)} \ge \frac{k}{\sinh(k)}$$

Suppose that $y \in C[0,1]$, $\sinh(k) - \beta\sinh(k\eta) > 0$ holds.Then the following linear boundary value proble

$$\begin{cases} -u^{''} + k^2 u = y(t) \ , t \in [0,1] \\ \qquad u(0) = 0, u(1) = \beta u(\eta) \end{cases}$$

has a unique positive solution u and u can be expressed in the form

$$u(t) \quad = \quad \int_0^1 G_1(t,s)y(s)ds \tag{2.15}$$

Where

$$G_1(t,s) = H_1(t,s) + \frac{\beta\sinh(kt)}{\sinh(k) - \beta\sinh(k\eta)} H_1(\eta,s) \tag{2.16}$$

Proof. We suppose the solution of the boundary value problem can be expressed by

$$u(t) = w(t) + A_1\sinh(kt) + A_2\sinh(k(1-t))$$

Than,by $u(0) = 0$,$u(1) = \beta u(\eta)$ we get :

$$A_2 \equiv 0, A_1 = \frac{\beta w(\eta)}{\sinh(k) - \beta\sinh(k\eta)}$$

2.2.1 Lemma

$G_1(t,s)$ has the following propertie

- $G_1(t,s) \geq 0$, $\forall t,s \in [0,1]$
- $\forall t,s \in [0,1]$, $\exists C_1, d_1 > 0$ such that

$$d_1 I(t)I(s) \leq G_1(t,s) \leq C_1 I(s) \quad \forall t,s \in [0,1]$$

where $I(s) = s(1-s)$

Proof. It is obvious that $G_1(t,s)$ is nonnegative.We have $H_1(t,s) \geq 0$ and $\sinh(k) - \beta \sinh(k\eta) > 0$.Then we suppose that $v(t) = \sinh(k)t - \sinh(kt), t \in [0,1]$ then

$v(0) = v(1) = 0$ and $v''(t) = -k^2 \sinh(kt) \leq 0, t \in [0,1]$. So $v(t) \geq 0$,i.e.

$$\sinh(k)t \geq \sinh(kt), t \in [0,1] \tag{2.17}$$

Similarly

$$kt \leq \sinh(kt), t \in [0,1] \tag{2.18}$$

we have

$$\frac{k}{\sinh(k)} H_1(t,t)H_1(s,s) \leq H_1(t,s) \leq H_1(t,t) \tag{2.19}$$

By using (10),(11),(12) and (13) we obtain

$$H_1(t,t) \geq \frac{(kt)(k(1-t))}{k\sinh(k)} = \frac{kI(t)}{\sinh(k)} \tag{2.20}$$

and

$$H_1(t,t) \leq \frac{(\sinh(k)t)(\sinh(k)(1-t))}{k\sinh(k)} = \frac{\sinh(k)I(t)}{k} \tag{2.21}$$

From (11),(13),(14)and (15) we have

$$G_1(t,s) \geq H_1(t,s) \geq \frac{k}{\sinh(k)} H_1(t,t)H_1(s,s) \geq (\frac{k}{\sinh(k)})^3 I(t)I(s)$$

and

$$G_1(t,s) \leq H_1(s,s) + H_1(s,s)\frac{\beta\sinh(ks)}{\sinh(k) - \beta\sinh(k\eta)}$$

$$G_1(t,s) \leq \frac{\sinh k}{k} I(s)[1 + \frac{\beta\sinh(k)}{\sinh(k) - \beta\sinh(k\eta)}]$$

Letting

$$C_1 = \frac{\sinh k}{k}[1 + \frac{\beta\sinh(k)}{\sinh(k) - \beta\sinh(k\eta)}]$$

and

$$d_1 = (\frac{k}{\sinh(k)})^3$$

we have

$$d_1 I(t)I(s) \leq G_1(t,s) \leq C_1 I(s) \quad \forall t,s \in [0,1]$$

2.3 Example 4

In this section we introduce some necessary definitions, lemmas, and preliminary results that are used in the main results which give the existence of solutions of the problem main-boundary. First, we construct Green's function for the linear boundary value problem

$$\begin{cases} u^{(4)}(t) + e(t) = 0, \quad 0 < t < +\infty\,; \\ u(0) = A,\ u'(0) = B,\ u''(t) - au'''(t) = \theta(t), \quad -\tau \le t \le 0\,; \\ u'''(+\infty) = C. \end{cases} \tag{2.22}$$

Let $e \in L^1[0, +\infty)$. Then the solution $u \in \mathcal{C}^3[-\tau, +\infty) \cap \mathcal{C}^4(0, +\infty)$ of the problem (**??**) can be expressed as

$$u(t) = \begin{cases} \phi(t), & -\tau \le t \le 0; \\ A + Bt + (aC + \theta(0))\dfrac{t^2}{2} + C\dfrac{t^3}{3!} + \displaystyle\int_0^\infty G(t,s)e(s)ds, & 0 \le t < +\infty, \end{cases} \tag{2.23}$$

where

$$G(t,s) = \begin{cases} \dfrac{a}{2}t^2 + \dfrac{st^2}{2} - \dfrac{s^2 t}{2} + \dfrac{s^3}{3!}, & 0 \le s \le t < +\infty\,; \\ \dfrac{a}{2}t^2 + \dfrac{t^3}{3!}, & 0 \le t \le s < +\infty, \end{cases} \tag{2.24}$$

and

$$\phi(t) = A + Bt + \left(\theta(0) + aC + a\int_0^\infty e(s)ds\right)\left(-at - a^2 + a^2 e^{\frac{t}{a}}\right) + \int_t^0 \left(s - a - t + ae^{\frac{t-s}{a}}\right)\theta(s)ds.$$

Proof. Since $e \in L^1[0, +\infty)$,we can integrate (**??**) from t to $+\infty$, and use $u'''(+\infty) = C$, to get

$$u'''(t) = C + \int_t^\infty e(s)ds, \quad t \ge 0.$$

Integrating the above equation on $[0, t]$, and applying Fubini's theorem and using $u''(0) - au'''(0) = \theta(0)$, we obtain

$$u''(t) = aC + a\int_0^\infty e(s)ds + \theta(0) + Ct + \int_0^t se(s)ds + \int_t^\infty te(s)ds. \tag{2.25}$$

Again integrating the equation (**??**) two times on $[0, t]$, and applying Fubini's theorem and using $u(0) = A$ and $u'(0) = B$, we find

$$u(t) = A + Bt + (aC + \theta(0))\frac{t^2}{2} + C\frac{t^3}{3!} + \int_0^t \left(\frac{a}{2}t^2 + \frac{st^2}{2} - \frac{s^2 t}{2} + \frac{s^3}{3!}\right)e(s)ds + \int_t^\infty \left(\frac{a}{2}t^2 + \frac{t^3}{3!}\right)e(s)ds,$$

for $t \in [0, +\infty)$. Now we consider the following third order linear differential equation

$$u''(t) - au'''(t) = \theta(t), \quad t \in [-\tau, 0].$$

If the above equation is rearranged, we have

$$u'''(t) - \frac{1}{a}u''(t) = -\frac{1}{a}\theta(t), \quad t \in [-\tau, 0],$$

and solve this linear equation on $[t, 0]$, we find

$$u''(t) = u''(0)e^{\frac{t}{a}} + \frac{1}{a}\int_t^0 e^{\frac{t-s}{a}}\theta(s)ds. \tag{2.26}$$

Next, integrating the equation (**??**) two times on $[t, 0]$, and applying Fubini's theorem and using the following boundary conditions

$$u(0) = A, \quad u'(0) = B \quad \text{and} \quad u''(0) = \theta(0) + au'''(0) = \theta(0) + aC + a\int_0^\infty e(s)ds,$$

we obtain

$$u(t) = A + Bt + \left(\theta(0) + aC + a\int_0^\infty e(s)ds\right)\left(-at - a^2 + a^2e^{\frac{t}{a}}\right) + \int_t^0 \left(s - a - t + ae^{\frac{t-s}{a}}\right)\theta(s)ds$$

for $t \in [-\tau, 0]$. This completes the proof of the lemma.

2.3.1 Remark

$G(t, s)$ defined in (**??**) is the Green's function of the BVP

$$\begin{cases} -u^{(4)}(t) = e(t), \ 0 < t < +\infty; \\ u(0) = u'(0) = 0, \ u''(0) = au'''(0), \ u'''(+\infty) = 0. \end{cases}$$

2.3.2 Properties of Green's function

The Green's function $G(t, s)$ has the following proprieties

1. $G(t, s)$ is two-times continuously differentiable on $[0, +\infty) \times [0, +\infty)$ and
$$\left.\frac{\partial^3 G(t,s)}{\partial t^3}\right|_{t=s^+} - \left.\frac{\partial^3 G(t,s)}{\partial t^3}\right|_{t=s^-} = -1;$$
2. $\dfrac{\partial^i G(t,s)}{\partial t^i} \geq 0, \quad \forall (t,s) \in [0,+\infty) \times [0,+\infty), \ for \ i = 0, 1, 2, 3;$
3. $\sup\limits_{t\in[0,+\infty)} \dfrac{G(t,s)}{1+t^3} \leq \left(\dfrac{a\sqrt[3]{4}+1}{6}\right), \quad \sup\limits_{t\in[0,+\infty)} \left(\dfrac{1}{1+t^2}\dfrac{\partial G(t,s)}{\partial t}\right) \leq \left(\dfrac{a+1}{2}\right),$
$\sup\limits_{t\in[0,+\infty)} \left(\dfrac{1}{1+t}\dfrac{\partial^2 G(t,s)}{\partial t^2}\right) \leq (a+1), \quad \sup\limits_{t\in[0,+\infty)} \dfrac{\partial^3 G(t,s)}{\partial t^3} \leq 1.$

Proof. (1) and (2) are obvious. Here we shall prove the first inequality of (3). We note that for all integers k and l

$$\sup_{t\in[0,+\infty)} \frac{t^k}{1+t^l} = \begin{cases} \frac{l-k}{l}\left(\frac{k}{l-k}\right)^{\frac{k}{l}}, & k < l\,; \\ 1, & k = l\,; \\ +\infty, & k > l. \end{cases}$$

For $s \le t$, we have

$$\begin{aligned}\sup_{t\in[0,+\infty)} \frac{G(t,s)}{1+t^3} &= \sup_{t\in[0,+\infty)} \left(\frac{\frac{a}{2}t^2 + \frac{st^2}{2} - \frac{s^2t}{2} + \frac{s^3}{6}}{1+t^3}\right) \le \sup_{t\in[0,+\infty)} \left(\frac{\frac{at^2}{2}}{1+t^3} + \frac{\frac{t^3}{6}}{1+t^3}\right)\\ &\le \frac{a}{2}\sup_{t\in[0,+\infty)} \frac{t^2}{1+t^3} + \frac{1}{6}\sup_{t\in[0,+\infty)} \frac{t^3}{1+t^3} \le \frac{a\sqrt[3]{4}+1}{6},\end{aligned}$$

and for $s \ge t$

$$\begin{aligned}\sup_{t\in[0,+\infty)} \frac{G(t,s)}{1+t^3} &= \sup_{t\in[0,+\infty)} \left(\frac{\frac{a}{2}t^2 + \frac{t^3}{6}}{1+t^3}\right) \le \sup_{t\in[0,+\infty)} \left(\frac{\frac{at^2}{2}}{1+t^3} + \frac{\frac{t^3}{6}}{1+t^3}\right)\\ &\le \frac{a}{2}\sup_{t\in[0,+\infty)} \frac{t^2}{1+t^3} + \frac{1}{6}\sup_{t\in[0,+\infty)} \frac{t^3}{1+t^3} \le \frac{a\sqrt[3]{4}+1}{6}.\end{aligned}$$

The other parts can be proved similarly.

We consider the space X defined by

$$X = \left\{u \in C^3[-\tau,+\infty) : \quad \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t^3} < +\infty, \sup_{t\in[0,+\infty)} \frac{|u'(t)|}{1+t^2} < +\infty, \right.$$
$$\left. \sup_{t\in[0,+\infty)} \frac{|u''(t)|}{1+t} < +\infty, \lim_{t\to+\infty} u'''(t) \text{ exists}\right\}$$

with the norm

$$\|u\| = \max\{\|u\|_0, \|u\|_1, \|u\|_2, \|u\|_\infty^0, \|u\|_\infty^1, \|u\|_\infty^2, \|u\|_\infty^3\}$$

where

$$\begin{aligned}\|u\|_0 &= \max_{t\in[-\tau,0]} |u(t)|, \quad \|u\|_\infty^0 = \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t^3},\\ \|u\|_1 &= \max_{t\in[-\tau,0]} |u'(t)|, \quad \|u\|_\infty^1 = \sup_{t\in[0,+\infty)} \frac{|u'(t)|}{1+t^2},\\ \|u\|_2 &= \max_{t\in[-\tau,0]} |u''(t)|, \quad \|u\|_\infty^2 = \sup_{t\in[0,+\infty)} \frac{|u''(t)|}{1+t}, \quad \|u\|_\infty^3 = \sup_{t\in[-\tau,+\infty)} |u'''(t)|.\end{aligned}$$

It is clear that $(X, \|.\|)$ is a Banach space. Next we define the mapping $T : X \to \mathcal{C}^3[-\tau,+\infty) \cap \mathcal{C}^4(0,+\infty)$ by

$$Tu(t) = \begin{cases} \psi(t), & -\tau \le t \le 0;\\ l(t) + \displaystyle\int_0^\infty G(t,s)q(s)f(s,[u(s)],[u'(s)],[u''(s)],u'''(s))ds, & 0 \le t < +\infty,\end{cases} \tag{2.27}$$

where

$$\begin{aligned}\psi(t) &= A + Bt + \left(\theta(0) + aC + a\int_0^\infty q(s)f(s,[u(s)],[u'(s)],[u''(s)],u'''(s))ds\right)\left(-at - a^2 + a^2 e^{\frac{t}{a}}\right)\\ &\quad + \int_t^0 \left(s - a - t + ae^{\frac{t-s}{a}}\right)\theta(s)ds\end{aligned}$$

and

$$l(t) = A + Bt + (aC + \theta(0))\frac{t^2}{2} + C\frac{t^3}{3!}. \tag{2.28}$$

2.3.3 Lemma

The mapping $T : X \to \mathcal{C}^3[-\tau, +\infty) \cap \mathcal{C}^4(0, +\infty)$ in (**??**) has the following properties :

1. $Tu(0) = A,\ (Tu)'(0) = B,\ (Tu)''(t) - a(Tu)'''(t) = \theta(t)$ for $t \in [-\tau, 0]$,
2. $Tu(t)$ is three-times continuously differentiable on $t \in [-\tau, +\infty)$,
3. $(Tu)^{(4)}(t) = -q(t)f(t, [u(t)], [u'(t)], [u''(t)], u'''(t)),\ t \in (0, +\infty)$,
4. fixed points of T are solutions of BVP (**??**)-(**??**).

When applying the Schäuder fixed point theorem to prove the existence result, it is necessary to show that the operator T_1 (defined later) is completely continuous. For this, we need the following modified version of the Arzela-Ascoli lemma (see [18,20]).

2.3.4 lemma

$M \subset X$ is relatively compact if the following conditions hold :

1. all functions belonging to M are uniformly bounded,
2. all functions belonging to M are equi-continuous on any compact sub-interval of $[-\tau, +\infty)$,
3. all functions from M are equi-convergent at infinity, that is, for any $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that for all $t \geq T$ and any $u \in M$,

$$\left| \frac{u(t)}{1+t^3} - \lim_{t\to+\infty} \frac{u(t)}{1+t^3} \right| < \epsilon,\ \left| \frac{u'(t)}{1+t^2} - \lim_{t\to+\infty} \frac{u'(t)}{1+t^2} \right| < \epsilon,$$
$$\left| \frac{u''(t)}{1+t} - \lim_{t\to+\infty} \frac{u''(t)}{1+t} \right| < \epsilon \ and\ |u'''(t) - \lim_{t\to+\infty} u'''(t)| < \epsilon.$$

Chapitre 3

Existence and nonexistence of positive solutions for singular nth-order three-point nonhomogeneous boundary value problem

3.1 Introduction

The importance of boundary-value problems in a wide variety of applications in the physical, biological and engineering sciences is now well documented in the literature, and in the years this has become an extremely active area of research. The monographs of Agarwal [?] and Agarwal, O'Regan, and Wong [?] contain excellent surveys of knows results. there is currently a great deal of interest in positive solutions for several types of boundary value problems. Some works have been done on nonlinear nth-order boundary value problems. For instance, R.Graff, J.Henderson and B.Yang [?], investigated the nth-order boundary value problems using the Krasnosl'skii fixed point boundary [?]

value problem :

$$\begin{cases} u^{(n)}(t) = \; g(t)f(u(t)), & 0 < t < 1, \\ u^{(i-1)}(0) = 0, & 1 \leq (i) \leq (n-2), \\ u^{(n-2)}(p) = 0, u^{(n-1)}(1) = 0. & \end{cases}$$

where $n \geq 4$ is fixed integer and $p \in (\frac{1}{2}, 1)$.

In [?], J. Henderson and S.Ntouyas investigated the existence of positive solutions for the the system of nonlinear differential equation :

$$\begin{cases} u^{(n)} + \lambda a(t)f(v) = 0, \quad v^{(n)} + \lambda b(t)f(u) = 0, \\ u(0) = u'(0) = 0 = ... = u^{(n-2)}(0) = 0, \quad u(1) = \alpha u(\eta) \\ v(0) = v'(0) = 0 = ... = v^{(n-2)}(0) = 0, \quad v(1) = \alpha u(\eta) \end{cases}$$

where $0 < \eta < 1$ is fixed integer, $0 < \alpha\eta^{n-1} < 1$.

In [?], by making use of the fixed point theorem and degree theory, El Shahed studied the existence of positive solutions for the following nth-order two-point boundary value problems :

$$\begin{cases} u^{(n)}(t) + \lambda a(t) f(t, u(t)) = 0, \\ u(0) = u''(0) = 0 = ... = u^{(n-1)}(0) = 0, \quad u'(1) = 0, \\ u(0) = u'(0) = u''(0) = 0 = ... = u^{(n-2)}(0) = 0, \quad u'(1) = 0, \\ u(0) = u'(0) = u''(0) = 0 = ... = u^{(n-2)}(0) = 0, \quad u''(1) = 0. \end{cases}$$

In [?], Y. Sun consider the following third-order boundary value problems :

$$\begin{cases} u'''(t) + a(t) f(t, u(t)) = 0, \\ u(0) = u'(0) = 0, \\ u'(1) - \alpha u'(\eta) = \lambda. \end{cases}$$

where $0 < \eta < 1$ and $\alpha \in [0, \frac{1}{\eta})$ are constants and $\lambda \in [0, +\infty)$ is parameter.

The author obtained the existence and nonexistence of positive solutions by applying the Guo-Krasnosel'skii's fixed point theorem and Schauder's fixed point theorem.

Odda [?], studied the nth-order boundary value problems using the Krasnosl'skii fixed Point for the problem of nonlinear differential equation :

$$\begin{cases} u^{(n)}(t) = f(t, u(t)), \\ u(0) = u'(0) = u''(0) = 0 = ... = u^{(n-2)}(0)) = u^{(n-1)}(0) = 0, \\ \alpha u'(1) + \beta u''(1) = 0. \end{cases}$$

where $\alpha, \beta \geq 0$, $\alpha + \beta > 0$.

In [?], by applying the Krasnosel'skii's fixed point Sun and Zhu established the existence and nonexistence of positive solutions for the following fourth-order three point boundary value problem.

$$\begin{cases} u^{(4)} + f(t, u(t)) = 0, & 0 < t < 1, \\ \\ u(0) = u'(0) = u''(0) = 0, & u''(1) - \alpha u''(\eta) = \lambda. \end{cases}$$

where the nonlinear term $f(t, u)$ may be singular at $t = 0$, $t = 1$.

Inspired and motivated by the works mentioned above, we deal with the existence of positive solutions for the nonlinear boundary value problem :

$$\begin{cases} u^{(n)}(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ \\ u(0) = u'(0) = u''(0) = 0 = ... = u^{(n-2)}(0) = 0, \\ \\ u^{(n-2)}(1) - \alpha u^{(n-2)}(\eta) = \lambda. \end{cases} \tag{3.1}$$

where $0 < \eta < 1$, $\alpha \in [0, \frac{1}{\eta})$ are constants, $\lambda \in [0, +\infty)$ is a parameter, $f(t, u(t))$ may be singular at $t = 0$ and $t = 1$. Here by a positive solution we mean a function $u^*(t)$ which is positive on $(0, 1)$ and satisfies Problem (??). To our best knowledge, no paper has considered Problem (??). The arguments are based upon the fixed point theorem for the special cone. The paper is organized as follows. The second section, we give some properties of Green's function associated with Problem (??) and construct a suitable cone and transforms Problem (??) into an integral equation. The third section, we discuss the existence and nonexistence of at least one positive solution for Problem (??). Finally we give two examples to illustrate ours results in fourth section.

3.2 Preliminaries

[Guo-Krasnosel'skii's Fixed Point Theorem [?, ?]] Let $(E, \|.\|)$ be a Banach space and $K \subset E$ be cone in E. Assume that Ω_1 and Ω_2 are open with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$,and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ be a completely continuous operator such that either :

(i) $\|Tu\| \leq \|u\|$,$\forall u \in K \cap\ \Omega_1$ and $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_2$

(ii) $\|Tu\| \geq \|u\|$, $\forall u \in K \cap\ \Omega_1$ and $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_2$,

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Suppose that $p(t) \in L^1(0, 1)$ and $p(t) > 0$. Then the linear boundary value problem

$$\begin{cases} u^{(n)}(t) + p(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, ..., u^{(n-2)}(0) = 0, & \\ u^{(n-2)}(1) - \alpha u^{(n-2)}(\eta) = \lambda. & \end{cases} \tag{3.2}$$

has a unique solution, which can be expressed by

$$u(t) = \int_0^1 G(t, s)p(s)ds + \frac{\alpha t^{n-1}}{(n-1)!(1-\alpha\eta)} \int_0^1 K(\eta, s)p(s)ds + \frac{\lambda t^{n-1}}{(n-1)!(1-\alpha\eta)}$$

where

$$G(t, s) = \begin{cases} \frac{t^{n-1}(1-s)-(t-s)^{n-1}}{(n-1)!}, & 0 \leq s \leq t \leq 1, \\ \\ \frac{t^{n-1}(1-s)}{(n-1)!}, & 0 \leq t \leq s \leq 1. \end{cases}$$

and

$$K(t, s) = \frac{\partial^{n-2}}{\partial t^{n-2}} G(t, s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof. In fact, if $u(t)$ is a solution of Problem (??) by the Taylor expansion formula, we have

$$u(t) = \sum_{0 \leq i \leq n-2} C_i t^i + \frac{C_{n-1}}{(n-1)!} t^{n-1} - \frac{1}{(n-1)!} \int_0^t (t-s)p(s)ds$$

which together with the boundary condition implies $C_i = 0\ \forall i = 0, 1, ..., n-2$.

$$C_{n-1} = \frac{1}{1-\alpha\eta}\int_0^1 (1-s)p(s)ds - \frac{\alpha}{1-\alpha\eta}\int_0^\eta (\eta-s)p(s)ds + \frac{\lambda}{1-\alpha\eta}$$

Therefore

$$\begin{aligned}
u(t) = &-\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}p(s)ds + \frac{t^{n-1}}{(n-1)!(1-\alpha\eta)}\int_0^1 (1-s)p(s)ds \\
&-\frac{\alpha t^{n-1}}{(n-1)!(1-\alpha\eta)}\int_0^\eta (\eta-s)p(s)ds + \frac{\lambda t^{n-1}}{(n-1)!(1-\alpha\eta)} \\
&= \frac{1}{(n-1)!}\int_0^t [t^{n-1}(1-s)-(t-s)^{n-1}]p(s)ds + \frac{t^{n-1}}{(n-1)!}\int_t^1 (1-s)p(s)ds \\
&+\frac{\alpha t^{n-1}}{(n-1)!(1-\alpha\eta)}[\int_0^\eta s(1-\eta)p(s)ds + \int_\eta^1 \eta(1-s)p(s)ds] + \frac{\lambda t^{n-1}}{(n-1)!(1-\alpha\eta)} \\
&= \int_0^1 G(t,s)p(s)ds + \frac{\alpha t^{n-1}}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)ds + \frac{\lambda t^{n-1}}{(n-1)!(1-\alpha\eta)}.
\end{aligned}$$

The proof is complete.

For all $(t,s) \in [0,1]\times[0,1]$, we have

$$\frac{1}{(n-1)!}t^{n-1}s(1-s) \le G(t,s) \le s(1-s)$$

Proof. It is obvious that $G(t,s) \ge 0$ and $G(t,s) \le G(1,s)$.

If $0 \le t \le s \le 1$, then

$$G(t,s) = \frac{1}{(n-1)!}t^{n-1}(1-s) \le \frac{1}{(n-1)!}s^{n-1}(1-s) \le s(1-s)$$

and

$$G(t,s) = \frac{1}{(n-1)!}t^{n-1}(1-s) \ge \frac{1}{(n-1)!}t^{n-1}s(1-s)$$

If $0 \le s \le t \le 1$, then

$$\begin{aligned}
G(t,s) &= \frac{1}{(n-1)!}\left[(1-s)t^{n-1} - (t-s)^{n-1}\right] \\
&\le \frac{s}{(n-1)!}\left[(1-s)t^{n-2} + 2t^{n-3}(1-s)\right] \\
&\le \frac{s(1-s)}{(n-1)!}(t^{n-2}+2t^{n-3}) \\
&\le s(1-s)
\end{aligned}$$

and

$$\begin{aligned}
G(t,s) &= \frac{1}{(n-1)!}\Big[(1-s)t^{n-1} - (t-s)^{n-1}\Big] \\
&\geq \frac{1}{(n-1)!}\Big[(1-s)t^{n-1} - (t-ts)^{n-1}\Big] \\
&\geq \frac{t^{n-1}}{(n-1)!}\Big[(1-s) - (1-s)^{n-1}\Big] \\
&\geq \frac{t^{n-1}}{(n-1)!}(1-s)\Big[1 - (1-s)^{n-2}\Big] \\
&\geq \frac{t^{n-1}}{(n-1)!}s(1-s)
\end{aligned}$$

3.3 Main Results

In this section, we shall give sufficient conditions on λ and f such that positive solutions with respect a cone for our Problem (**??**) exist. We shall investigate the nonexistence of positive solutions of our problem. We present the assumptions that we shall use in the sequel. Let $E = C[0,1]$ be a Banach space of all continuous fonctions with the norm

$$\|u\| = \max_{0\leq t\leq 1} |u(t)| \quad C^+[0,1] = \{u \in C[0,1] : u(t) > 0, t \in [0,1]\}$$

Throughout the paper, we assume that

(H_1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous

(H_2) There exists a continuous function $q : (0,1) \to [0,\infty)$ such that

$$0 < \int_a^b s(1-s)q(s)ds \leq \int_0^1 s(1-s)q(s)ds < \infty. \qquad for \quad [a,b] \subset [0,1]$$

(H_3) There exists a continuous function $h : [0,1] \times [0,\infty) \to [0,\infty)$ such that

$$f(t,u) \leq q(t)h(t,u), \qquad (t,u) \in (0,1) \times [0,\infty)$$

Define a cone $K \subset C[0,1]$ by

$$K = \{u(t) \in C^+[0,1] : u(t) \geq \frac{1}{(n-1)!}t^{n-1}\|u\|, 0 \leq t \leq 1\}$$

Then K is positive cone in $C[0,1]$. Denote $\Omega_r = \{u \in K : \|u| \ < r\}$, $\partial\Omega_r = \{u \in K : \|u\| = r\}$ Fix $R > r$. Define an operator $T : (\overline{\Omega_R}\backslash\Omega_r) \cap K \to K$ by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds + \frac{\alpha t^{n-1}}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)f(s,u(s))ds + \frac{\lambda t^{n-1}}{(n-1)!(1-\alpha\eta)}$$

It is well know that the Problem (**??**) has a positive solution $u(t)$ if and only if u is a fixed point of T.

Suppose $(H_1) \sim (H_3)$ hold. Then $T(K) \subseteq K$ **Proof.** From (H_2) and (H_3), we know that

$$\begin{aligned}
&0 \leq (Tu)(t)\\
&\leq \int_0^1 s(1-s)f(s,u(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)f(s,u(s))ds + \frac{\lambda}{(n-1)!(1-\alpha\eta)}\\
&\leq \int_0^1 s(1-s)q(s)h(s,u(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta)q(s)h(s,u(s))ds + \frac{\lambda}{(n-1)!(1-\alpha\eta)} < +
\end{aligned}$$

On the other hand, for any $u \in K$ we have $u(t) \geq \frac{1}{(n-1)!}t^{n-1}\|u\|$, $t \in [0,1]$ and

$$\|Tu\| \leq \int_0^1 s(1-s)f(s,u(s))ds + \frac{\alpha}{(1-\alpha\eta)}\int_0^1 K(\eta,s)f(s,u(s))ds + \frac{\lambda}{(1-\alpha\eta)}$$

Therefore

$$\begin{aligned}
(Tu)(t) &\geq \frac{1}{(n-1)!}t^{n-1}\Big[\int_0^1 s(1-s)f(s,u(s))ds + \frac{\alpha}{(1-\alpha\eta)}\int_0^1 K(\eta,s)f(s,u(s))ds + \frac{\lambda}{(1-\alpha\eta)}\Big]\\
&\geq \frac{1}{(n-1)!}t^{n-1}\|Tu\|.
\end{aligned}$$

The proof is complete.

Suppose that $(H_1) \sim (H_3)$ hold. Then $T : (\overline{\Omega_R}\backslash\Omega_r) \cap K \to K$ is completely continuous. **Proof.** For any $u \in (\overline{\Omega_R}\backslash\Omega_r) \cap K$, we have $0 \leq \frac{1}{(n-1)!}t^{n-1}r \leq \frac{1}{(n-1)!}t^{n-1}\|u\| \leq u(t) \leq R$

Let $(u_n(t))$ converge to $u(t)$. Then

$$\begin{aligned}
|Tu_n(t) - Tu(t)| = \quad & \int_0^1 G(t,s)(f(s,u_n(s)) - f(s,u(s))ds +\\
& \frac{\alpha t^{n-1}}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)(f(s,u_n(s)) - f(s,u(s))ds\\
& \leq \max_{0\leq t\leq 1}\int_0^1 G(t,s)q(s)\big(h(s,u_n(s)) - h(s,u(s))\big)ds\\
& + \max_{0\leq t\leq 1}\Big\{\frac{\alpha t^{n-1}}{(n-1)!(1-\alpha\eta)}\Big\}\int_0^1 K(\eta,s)q(s)\big(h(s,u_n(s)) - h(s,u(s))\big)ds\\
& \leq \max_{0\leq t\leq 1}\int_0^1 G(t,s)q(s)\|h(s,u_n(s)) - h(s,u(s))\|ds\\
& + \max_{0\leq t\leq 1}\Big\{\frac{\alpha t^{n-1}}{(n-1)!(1-\alpha\eta)}\Big\}\int_0^1 K(\eta,s)q(s)\|h(s,u_n(s)) - h(s,u(s))\|ds
\end{aligned}$$

Further, $G(t,s)$ and $K(t,s)$ are continuous in t so the Lebesgue dominated convergence theorem implies that the operator T is continuous. Suppose that $B \subset K$ is a bounded set, then there exist $b > 0$ such that $\|u\| \leq b$ for any

$u \in B$. From (H_3) we know that $|f(t,u)| \leq q(t)h(t,u) \leq M(b)q(t)$, for $(t,u) \in (0,1)\times[0,b]$, where $M(b) = \max\{h(t,u); (t,u) \in (0,1)\times[0,b]\}$ .Then, we have

$$\begin{aligned} \|Tu\| &\leq \int_0^1 s(1-s)f(s,u(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)f(s,u(s))ds + \frac{\lambda}{(n-1)!(1-\alpha\eta)} \\ &\leq \int_0^1 s(1-s)q(s)h(s,u(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)h(s,u(s))ds + \frac{\lambda}{(n-1)!(1-\alpha\eta)} \\ &\leq M(b)\big(\int_0^1 s(1-s)q(s)ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)ds\big) + \frac{\lambda}{(n-1)!(1-\alpha\eta)} < +\infty \end{aligned}$$

Hence T is uniformly bounded.

On the other hand, let $t_1, t_2 \in [0,1]$, $t_1 < t_2$. Then we have

$$\begin{aligned} |(Tu)(t_1)-(Tu)(t_2)| &= \frac{1}{(n-1)!}\int_0^{t_1}(t_2-s)^{n-1}-(t_1-s)^{n-1}|f(s,u(s))|ds \\ &+\frac{1}{(n-1)!}\int_{t_1}^{t_2}(t_2-s)^{n-1}|f(s,u(s))|ds + \frac{(t_2^{n-1}-t_1^{n-1})}{(n-1)!(1-\alpha\eta)}\int_0^1(1-s)|f(s,u(s))|ds \\ &+\frac{\alpha(t_2^{n-1}-t_1^{n-1})}{(n-1)!(1-\alpha\eta)}\int_0^\eta(\eta-s)|f(s,u(s))|ds + \frac{\lambda(t_2^{n-1}-t_1^{n-1})}{(n-1)!(1-\alpha\eta)} \end{aligned}$$

Let $\psi(t) = t^{n-1}-(n-1)t$. It is easy to see that the function ψ is nonincreasing on $[0,1]$. Immediately we have $(t_2-s)^{n-1}-(t_1-s)^{n-1} \leq (n-1)(t_2-t_1)$.

Therefore we get,

$$\begin{aligned} |(Tu)(t_1)-(Tu)(t_2)| &= \frac{1}{(n-2)!}(t_2-t_1)\int_0^{t_1}|f(s,u(s))|ds + \int_0^{t_2}|f(s,u(s))|ds \\ &+\frac{(t_2^{n-1}-t_1^{n-1})}{(n-2)!(1-\alpha\eta)}\big[\int_0^1|f(s,u(s))|ds + \frac{\lambda}{n-1}\big] \end{aligned}$$

$\longrightarrow 0$ as $t_1 \to t_2$.

It shows from using the Arzela-Ascoli Theorem , that T is equicontinuous. Consequently, T is completely continuous.

Suppose that (H_1), (H_2) and (H_3) hold.In addition assume that the following conditions hold.

(H_4) **:** $\limsup_{u\to 0^+}\max_{0\leq t\leq 1}\frac{h(t,u)}{u} = 0$

(H_5) **:** $\liminf_{u\to+\infty}\min_{a\leq t\leq b}\frac{f(t,u)}{u} = +\infty$

Then the Problem (**??**) has at least one positive solution for λ small enough, and Problem (**??**) has no positive solution for λ large enough.

Proof. For $\lambda > 0$ small enough, let

$$M = \big(1-\frac{1}{(n-1)!}\big)\Big[\int_0^1 s(1-s)q(s)ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)ds\Big]^{-1} \quad (3.3)$$

From (H_4), there exist $R_1 > 0$ such that $h(t,u) \leq Mu$ for $(t,u) \in [0,1] \times [0,R_1]$. Let $\Omega_1 = \{u \in K : \|u\| < R_1\}$, $0 < \lambda \leq (1-\alpha\eta)R_1$. For any $u \in K \cap \partial\Omega_1$, we get

$$\begin{aligned}
(Tu)(t) \quad &= \int_0^1 G(t,s)f(s,u(s))ds + \frac{\alpha t^{n-1}}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)f(s,u(s))ds + \frac{\lambda t^{n-1}}{(n-1)!(1-\alpha\eta)} \\
&\leq \int_0^1 s(1-s)q(s)h(s,u(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)h(s,u(s))ds + \frac{\lambda}{(n-1)!(1-\alpha\eta)} \\
&\leq M\int_0^1 s(1-s)q(s)u(s)ds + \frac{M\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)u(s)ds + \frac{R_1(1-\alpha\eta)}{(n-1)!(1-\alpha\eta)} \\
&\leq M\|u\|\int_0^1 s(1-s)q(s)ds + \frac{M\alpha\|u\|}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)ds + \frac{R_1}{(n-1)!} \\
&\leq M\|u\|\big[\int_0^1 s(1-s)q(s)ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)ds\big] + \frac{R_1}{(n-1)!} \\
&\leq \big(1 - \frac{1}{(n-1)!}\big)R_1 + \frac{R_1}{(n-1)!} = R_1 = \|u\|.
\end{aligned}$$

Therefore $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$.

In the other hand, let

$$N = \frac{(n-1)!}{a^{n-1}}\big[\int_a^b s(1-s)ds + \frac{\alpha}{(1-\alpha\eta)}\int_a^b K(\eta,s)ds\big]^{-1}. \tag{3.4}$$

From (H_5), there exist $R > 0$ such that $f(t,u) \geq Nu$ for $(t,u) \in [0,1] \times [R,\infty)$. Let $R_2 > \frac{(n-1)!R}{a^{n-1}} > R_1$ and let $\Omega_2 = \{u \in K : \|u\| < R_2\}$. For any $u \in K \cap \partial\Omega_2$ and $t \in [a,b]$ we have $u(t) \geq \frac{1}{(n-1)!}t^{n-1}\|u\| = \frac{1}{(n-1)!}a^{n-1}R_2 > R$ and

$$\begin{aligned}
(Tu)(1) \quad &= \int_a^b G(1,s)f(s,u(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_a^b K(\eta,s)f(s,u(s))ds + \frac{\lambda}{(n-1)!(1-\alpha\eta)} \\
&\geq \int_a^b \frac{1}{(n-1)!}s(1-s)f(s,u(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_a^b K(\eta,s)f(s,u(s))ds \\
&\geq \int_a^b \frac{1}{(n-1)!}s(1-s)Nu(s)ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_a^b K(\eta,s)Nu(s)ds \\
&\geq N\frac{a^{n-1}}{(n-1)!}\big[\int_a^b s(1-s)ds + \frac{\alpha}{(1-\alpha\eta)}\int_a^b K(\eta,s)ds\big]\|u\|
\end{aligned}$$

which means $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$

Thus, condition (i) of Lemma (**??**) is satisfied and so the Problem (**??**) has at least one positive solution.

Shall show that for λ large enough, the problem (**??**) has no positive solution. Otherwise, there exists. $0 < \lambda_1 < \lambda_2 < ... < \lambda_n < ...$ with $\lim_{n\to+\infty}\lambda_n = +\infty$ such that problem has positive solution $u_n(t)$, then we get

$$\begin{aligned} u_n(1) &= \int_0^1 G(1,s)f(s,u_n(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)f(s,u_n(s))ds + \frac{\lambda_n}{(n-1)!(1-\alpha\eta)} \\ &\geq \frac{\lambda_n}{(n-1)!(1-\alpha\eta)} \to +\infty \qquad as \qquad n \to +\infty \end{aligned}$$

Hence $\|u_n\| \to +\infty$ as $n \to +\infty$.

Again from (H_5), there exists $R > 0$ and $N > 0$ such $f(t,u) \geq 2(n-1)!Nu$ for $(t,u) \in [a,b]\times[R,\infty)$ where N is defined by (4). Let $R' > \dfrac{(n-1)!a^{n-1}}{R}$ such that $\|u\| \geq R'$. Thus, we get

$$\begin{aligned} \|u_n\| \geq u_n(1) &= \int_0^1 G(1,s)f(s,u_n(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)f(s,u_n(s))ds + \frac{\lambda_n}{(n-1)!(1-\alpha\eta)} \\ &\geq \int_a^b \frac{1}{(n-1)!}s(1-s)f(s,u_n(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_a^b K(\eta,s)f(s,u_n(s))ds \\ &\geq 2N\Big[\int_a^b s(1-s)u_n(s)ds + \frac{\alpha}{(1-\alpha\eta)}\int_a^b K(\eta,s)u_n(s)ds\Big] \\ &\geq 2N\frac{a^{n-1}}{(n-1)!}\Big[\int_a^b s(1-s)ds + \frac{\alpha}{(1-\alpha\eta)}\int_a^b K(\eta,s)ds\Big]\|u\| \\ &= 2\|u\| \end{aligned}$$

Which is a contradiction. So the Problem (**??**) has no positive solution. The proof is complete.

Suppose that (H_1), (H_2) and (H_3) hold. In addition assume that the following conditions hold.

(H_6) **:** $\limsup\limits_{u\to 0^+} \max\limits_{a\leq t\leq b} \dfrac{f(t,u)}{u} = +\infty$

(H_7) **:** $\liminf\limits_{u\to +\infty} \min\limits_{0\leq t\leq 1} \dfrac{h(t,u)}{u} = 0$

Then the Problem (**??**) has at least one positive solution for any $\lambda \in [0,+\infty)$

Proof. From (H_6) there exist constants $R_1 > 0$ and $N > 0$ such that $f(t,u) \geq (n-1)!Nu$ for $(t,u) \in [a,b]\times(0,R_1]$ where N is defined by (**??**). Let $\Omega_1 = \{u \in K : \|u\| < R_1\}$. Applying a similar argument to that used in Theorem 1 it follows that $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$.

On the other hand, from (H_7) , there exists a constant $R > 0$ such that $h(t,u) \leq Mu$ for $u \geq R$ where M is defined by (**??**). Choose R_2 where $R_2 \geq \max\{2R_1, \frac{M^*}{M}, R, \frac{\lambda}{1-\alpha\eta}\}$ such that $\max\limits_{0\leq t\leq 1} h(t,u) \leq M^*$ for $(t,u) \in [0,1]\times(0,R]$. Let $\Omega_2 = \{u \in K : \|u\| < R_2\}$, $0 < \lambda \leq (1-\alpha\eta)R_1$.

For any $u \in K \cap \partial\Omega_1$, we get

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds + \frac{\alpha t^(n-1)}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)f(s,u(s))ds + \frac{\lambda t^{n-1}}{(n-1)!(1-\alpha\eta)}$$

$$\leq \int_0^1 s(1-s)q(s)h(s,u(s))ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)h(s,u(s))ds + \frac{\lambda}{(n-1)!(1-}$$

$$\leq M* \int_0^1 s(1-s)q(s)ds + \frac{M\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)ds + \frac{\lambda}{(n-1)!(1-\alpha\eta)}$$

$$\leq MR_2\big[\int_0^1 s(1-s)q(s)ds + \frac{\alpha}{(n-1)!(1-\alpha\eta)}\int_0^1 K(\eta,s)q(s)ds\big] + \frac{R_2}{(n-1)!}$$

$$\leq \big(1-\frac{1}{(n-1)!}\big)R_2 + \frac{R_2}{(n-1)!} = R_2 = \|u\|$$

So $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$

Thus, condition (ii) of Lemma (**??**) is satisfied and so the Problem (**??**) has at least one positive solution.

3.4 Examples

We give some examples to illustrate the main results in the paper.

Consider the following boundary value problem

$$\begin{cases} u^{(5)}(t) + \frac{1}{t^{\mu_1}(1-t)^{\mu_2+1}}(u^{\frac{3}{2}}(t)+u(t))\sin^2(3u(t)) = 0, & 0<t<1, \\ u(0)=u'(0)=u''(0)=u'''(0)=0, & \\ u'''(1)-\frac{1}{6}u'''(\frac{1}{3}) = 1. & \end{cases} \tag{3.5}$$

If $\mu_1 < 2$ and $\mu_2 < 1$, the Problem (**??**) has at least one positive solution.

Let

$$q(t) = \frac{1}{t^{\mu_1}(1-t)^{\mu_2+1}}, \qquad h(t,u) = (u^{\frac{3}{2}}+u)\sin^2(3u)$$

Take $[a,b] = [\frac{1}{4},\frac{3}{4}]$. Notice for any fixed $t \in (0,1)$, that $f(t,u) \leq q(t)g(t,u)$ and

$0 < \int_0^1 s(1-s)q(s)ds < \infty$ for $\mu_1 < 2$ and $\mu_2 < 1$ and conditions $(H_1) \sim (H_3)$ are satisfied. The condition (H_4) and (H_5) follows from

$$\limsup_{u\to 0^+} \max_{0\leq t\leq 1} \frac{2(u^{\frac{3}{2}}+u)\sin^2 3u}{u} = 0$$

$$\liminf_{u\to +\infty} \min_{\frac{1}{3}\leq t\leq \frac{3}{4}} \frac{(u^{\frac{3}{2}}+u)\sin^2 3u}{t^{\mu_1}(1-t)^{\mu_2+1}u} = +\infty$$

Thus, from Theorem (**??**), the Problem (**??**) has at least one positive solution if $\mu_1 < 2$ and $\mu_2 < 1$.

Consider the following boundary value problem

$$\begin{cases} u^{(6)}(t) + \frac{1,901(1+\sqrt{u(t)})+\sin^2 u(t)}{\sqrt{t(1-t)}}, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = u'''(0) = 0, \\ u'''(1) - \frac{1}{3}u'''(\frac{1}{2}) = 1,902. \end{cases} \tag{3.6}$$

Let

$$q(t) = \frac{1}{\sqrt{t(1-t)}}, \qquad h(t,u) = 1,902(1+\sqrt{u}) + \sin^2 u$$

Take $[a,b] = [\frac{1}{4}, \frac{3}{4}]$. Notice for any fixed $t \in (0,1)$, that $f(t,u) \leq q(t)g(t,u)$ and

$0 < \int_0^1 s(1-s)q(s)ds < \infty$ for $\mu_1 < 2$ and $\mu_2 < 1$ and conditions $(H_1) \sim (H_3)$ are satisfied. The condition (H_6) and (H_7) follows immediately from

$$\limsup_{u\to 0^+} \max_{\frac{1}{3}\leq t\leq \frac{3}{4}} \frac{1,901(1+\sqrt{u(t)})+\sin^2 u(t)}{\sqrt{t(1-t)}u(t)} = +\infty$$

$$\liminf_{u\to +\infty} \min_{0\leq t\leq 1} \frac{1,901(1+\sqrt{u(t)})+\sin^2 u(t)}{u(t)} = 0$$

Thus, from Theorem (**??**), the Problem (**??**) has at least one positive solution.

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