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Intitulé

Normal Operators on Hilbert Spaces

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Introduction

The operators that arise naturally in models of the physical, biological, or economic world are almost always unbounded operators. That is, they do not act continuously from one Banach space X to itself. For example, this applies to the operator of differentiation on functions of a single variable: if one takes X small enough so that the operator is defined on the whole of X. However, mathematicians study the theory of bounded operators. As for the theory of unbounded operators, it is studied at the graduate level, we find that the detailed theory is that of bounded operators, for several educational reasons. Firstly, the theory of bounded operators is fairly straightforward, as it sidesteps the complex issues related to the domains of unbounded operators.

This memory master is devoted to the study of some basic properties of linear operators defined on Hilbert spaces.

In **Chapter 1**, we recall some basic notions from Hilbert space theory, such as Hilbert spaces, Cauchy-Schwarz inequality, orthogonality, decomposition of Hilbert spaces, Riesz Representation Theorem, isomorphisms of Hilbert spaces. We then turn to bounded linear operators on Hilbert spaces, their operator norms and the existence of adjoints. We define the notions and properties of operators on Hilbert spaces (such as unitary, isometry, self-adjoint, normal, orthogonal projections, etc.). You may take [refer to [8] and [7]] as general references for Lecture 1. **Chapter 2** is devoted to study some definitions and elementary properties of unbounded operators [refer to [13] and [15]]. In **Chapter 3** we treat the problem of normality for the product of two operators not necessarily bounded since the product of two normal operators does not imply that it is normal [see [3]].

Chapter 1

Bounded Operators

In this chapter, we begin with the definition of inner product and we give the definition of a Hilbert space, after that we describe some important classes of bounded linear operators on Hilbert spaces, containing self-adjoint operators, normal operators and unitary operators.

1.1 Definitions and Properties

Let H be a vector space over $\mathbb{C}$. We say that a function, denoted $< ., . >: H \times H \longrightarrow \mathbb{C}$ is an **inner product** on H if it satisfies the following properties:

1. $\forall x, y, z \in H$, $< x + y, z > = < \text{x, ,y} > + < x, , z >$
2. $\forall \lambda \in \mathbb{C}$, $\forall x, y \in H$, $< \lambda x, y > = \lambda < x, y >$.
3. $\forall x \in H$, $< x, x > \geq 0$, and $< x, x > = 0$ if $x = 0$
4. $\forall x, y \in H$ $< x, y > = \overline{< y, x >}$ (conjugate symmetry).

The couple $(H, < ., . >)$ is called a **pre-Hilbert space**.

Examples 1.1.1.

1. *Let* $H = \mathbb{C}^n$. *The function*

$$< ., . >: \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$
$$(x, y) \longmapsto \sum_{k=1}^{n} = x_k \overline{y_k}$$

is an inner product on H.

2. *Let $H = \mathcal{C}([0,1], \mathbb{C})$. The function*

$$\begin{aligned} <.,.>: H \times H \longrightarrow & \mathbb{C} \\ (f,g) \longmapsto & \int_0^1 f(x)\overline{g(x)}dx \end{aligned}$$

is an inner product on H.

3. *Let $H = \ell^2(\mathbb{C}) = \{x = (x_n)_{n\geq 0} \subset \mathbb{C} : \sum_{n\geq 0} |x_n|^2 < \infty\}$. The function*

$$\begin{aligned} <.,.>: H \times H \longrightarrow & \mathbb{C} \\ (x,y) \longmapsto & \sum_{n\geq 0} x_n\overline{y_n} \end{aligned}$$

is an inner product on H.

4. *Let $H = L^2(\mathbb{R}) = \{f : \mathbb{R} \longrightarrow \mathbb{C}$ measurable and $\int_{\mathbb{R}} |f(x)|^2 dx < \infty\}$. The function*

$$\begin{aligned} <.,.>: H \times H \longrightarrow & \mathbb{C} \\ (f,g) \longmapsto & \int_{-\infty}^{+\infty} f(x)\overline{g(x)}dx \end{aligned}$$

is an inner product on H.

Proposition 1.1.1. *If $<.,.>$ is an inner product on H, then the function $\|.\| \longrightarrow \mathbb{R}$ defined by $\|x\| = \sqrt{<x,x>}$ is a norm on H.*

Proposition 1.1.2. *If $(H, <.,.>)$ is a pre-Hilbert space, then*

$$\forall x, y \in H : | <x,y> | \leq \|x\| \; \|y\|$$

*called **Cauchy-Schwarz inequality**.*

Proof. Let $x, y \in H$. We want to prove that:

$$| <x,y> | \leq \|x\| \; \|y\|$$

We have :

$$\|\alpha x - y\|^2 \geq 0$$

where α is a real number, since the norm is always non-negative, by expanding on the previous point:

$$\|\alpha x - y\|^2 =< \alpha x - y, \alpha x - y >$$

we use the properties of the inner product, we get:

$$\alpha^2 < x, x > -2\alpha < x, y > + < y, y >$$

since:

$$\langle x, x\rangle = \|x\|^2$$

$$\langle y, y\rangle = \|y\|^2$$

we find that the equation becomes:

$$\alpha^2\|x\|^2 - 2\alpha < x, y > +\|y\|^2 \geq 0$$

we get a quadratic equation in α:

$$a\alpha^2 + b\alpha + c \geq 0$$

where : $a = \|x\|^2$, $b = -2 < x, y >$, $c = \|y\|^2$ we impose that the discriminant must be non-positive for the quadratic equation to hold for all values of α:

$$b^2 - 4ac \leq 0$$

We replace a,b and c with their corresponding values

$$(-2 < x, y >)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

$$4 < x, y >^2 \leq 4\|x\|^2\|y\|^2$$

$$< x, y >^2 \leq \|x\|^2\|y\|^2$$

After simplification, we find that :

$$| < x, y > | \leq \|x\|\|y\|$$

□

Property 1.1.1. *Let H be a pre-Hilbert space. Then*

$$< x, y >= 0, \forall y \in H \Longleftrightarrow x = 0$$

Definition 1.1.1. *Let H be a pre-Hilbert space. We say that x and y are orthogonal written $x \perp y$, if $< x, y >= 0$. The* ***orthogonal complement*** *of a subset $A \subset H$ is denoted and defined by*

$$A^{\perp} = \{x \in H :< x, a >= 0, \forall a \in A\}$$

Property 1.1.2. *Let H be a pre-Hilbert space and $A \subset H$, we have:*

1. $0 \in A^{\perp}$
2. *If $0 \in A$, then $A \cap A^{\perp} = \{0\}$, otherwise $A \cap A^{\perp} = \varnothing$*
3. $\{0\}^{\perp} = H$ *and* $H^{\perp} = \{0\}$.
4. *If $A \subset B$ then $B^{\perp} \subset A^{\perp}$*
5. *$A^{\perp}$ is a closed subspace of H.*
6. *$A \subset (A^{\perp})^{\perp}$ and $A^{\perp} = A^{\perp\perp\perp}$*

Definition 1.1.2. *A Hilbert space H is a complete pre-Hilbert space.*

Theorem 1.1.1. *Let H be a Hilbert space and let C be a closed and convex subset of H. Then for every point $p \in H$, there exists a unique point x_p such that*

$$\|p - x_p\| = inf\{\|p - x\| : a \in C\}.$$

This point x_p is called the ***orthogonal projection*** *of p onto C.*

Theorem 1.1.2. *Let H be a Hilbert space and let Y be a closed subspace of H, then:*

$$H = Y \oplus Y^{\perp}$$

Remark 1.1.1. *To show that A is dense in H (i.e $\overline{A} = H$), we show that $A^{\perp} = \{0\}$ as $A^{\perp\perp} = \{0\}^{\perp} = H$.*

Definition 1.1.3. *Let H be a Hilbert space and let $T : H \longrightarrow H$ be a linear function. We say that T is bounded (i.e. continuous) if*

$$\exists c > 0 : \forall x \in H, \|T(x)\| \leq c\|x\|.$$

We denote by $B(H)$ the space of all bounded operators $T : H \longrightarrow H$.

Examples 1.1.2.

1. *Let H be a Hilbert space. The identity operator $I : H \longrightarrow H$ is bounded as*

$$\forall x \in H : \|I(x)\| = \|x\|$$

2. *Let $H = \ell^2(\mathbb{C}) = \{(x_n)_{n\geq 0} \subset \mathbb{C} : \sum_{n\geq 0} |x_n|^2 < \infty\}$. The **shift** operator, defined by*

$$\begin{aligned} T : H \longrightarrow & H \\ x \longmapsto & T(x) = (0, x_1, x_2, x_3, ...) \end{aligned}$$

is bounded as

$$\forall x \in H : \|T(x)\| = \|x\|$$

3. *$H = \ell^2(\mathbb{C})$.Define the operator:*

$$\begin{aligned} T : H \longrightarrow & H \\ x \longmapsto & S(x) = (x_1, 3x_2, x_3, 3x_4, x_5, 3x_6, ...) \end{aligned}$$

we determine the norm squared.

$$\|Tx\|^2 = |x_1|^2 + 9|x_2|^2 + |x_3|^2 + 9|x_4|^2 + ... + 9|x_n|^2 + ...,$$

$$\|Tx\|^2 \leq 9|x_1|^2 + 9|x_2|^2 + 9|x_3|^2 + 9|x_4|^2 + ... + 9|x_n|^2 + ...,$$

Therefore,

$$\|Tx\|^2 \leq 9\|x\|^2$$

so

$$\exists c = 3 > 0 \quad \textit{such that} : \|Tx\| \leq 3\|x\| \quad , \forall x \in \ell^2.$$

This shows that T is a bounded operator.

4. *Let $H = \ell^2(\mathbb{C})$. The operator*

$$\begin{aligned} T : H \longrightarrow & H \\ x \longmapsto & T(x) = (x_1, 2x_2, 3x_3, ..., nx_n, ...) \end{aligned}$$

This operator is not bounded, and we can show this using a specific sequence. Let us define a sequence $(e_n)_{n\geq 1}$ in H by

$$e_n = (0, 0, 0, ..., \underbrace{1}_{n\text{-}th}, 0, 0, ...)$$

but $(Te_n)_{n\geq 1}$ is not bounded since

$$\|Te_n\| = \|(0, 0, 0, ..., n, 0, ..)\| = n\|e_n\|$$

Then

$$\|Te_n\| = n$$

So $\|Te_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$.

5. *Let $H = L^2[0, 1]$ and φ be a continuous function on $[0, 1]$. The operator*

$$T\varphi : H \longrightarrow H$$
$$f \longmapsto T_\varphi f = \varphi f$$

is bounded as

$$\|T_\varphi f\|^2 = \int_0^1 |\varphi(x)f(x)|^2 dx$$
$$\|T_\varphi f\|^2 \leq \|\varphi\|_\infty^2 \ \|f\|_{L^2}^2,$$

so $\exists c = \|\varphi\|_\infty$ such that $\|T_\varphi f\| \leq c\|f\|$, $\forall f \in L^2[0, 1]$.

Definition 1.1.4. *Let $T \in B(H)$. Set $\|T\| = inf\{c >, \forall x \in H : \|Tx\| \leq c\|x\|\}$. $\|T\|$ is called the **norm** of T.*

Definition 1.1.5. *Let $T \in B(H)$, We have:*

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=\|y\|=1} | < Tx, y > |.$$

Theorem 1.1.3 (Banach isomorphism)**.** *Let H be a Hilbert space and $T \in B(H)$. If T is bijective, then T^{-1} is bounded.*

Proof. Since H is a Hilbert space, it is also a Banach space (which means it is complete), given that $T \in B(H)$, so it is linear, bounded and bijective, so the inverse T^{-1} exists and is linear. By the Open Mapping Theorem. This theorem states that a bounded and surjective linear operator between Banach spaces is an open mapping, that is, it sends open sets to open sets. Since T satisfies these conditions, it is an open map. That means its inverse T^{-1} is continuous. Because T^{-1} is a linear and continuous operator on Banach space, it must also be bounded. So $T^{-1} \in B(H)$

□

Theorem 1.1.4 (Closed graph). *See[15] Let H and K be Banach spaces and $T : H \longrightarrow K$ be a linear transformation and let G_T be a graph of T. Then:*

$$T \text{ is continuous} \Longleftrightarrow G_T \text{ is closed in } H \times K$$

Theorem 1.1.5 (Riesz). *Let H be a $\mathbb{C}$ Hilbert and $\phi : H \longrightarrow \mathbb{C}$ be a bounded linear functional. Then there exists a unique $y \in H$ such that*

$$\phi(x) =< x, y >, \forall x \in H.$$

Proof. See[16] If $\phi = 0$, then $y = 0$, so we suppose that $\phi \neq 0$. In this case, $\ker \phi$ is a proper closed subspace of H. There exists a nonzero vector $z \in H$ such that $z \perp \ker \phi$. We define a linear map $P : H \to H$ by

$$Px = \frac{\phi(x)}{\phi(z)} z.$$

Thus $P^2 = P$, and we have $H = ranP \bigoplus \ker P$. Moreover,

$$ranP = \{\alpha z | \alpha \in \mathbb{C}\}, \ker P = \ker \phi,$$

So $ranP \perp \ker P$. Hence P is an orthogonal projection, and therefore

$$H = \{\alpha z | \alpha \in \mathbb{C}\} \oplus \ker \phi$$

is an orthogonal direct sum. Consequently, any vector

$$x \in H \text{can be written as } x = \alpha z + n, \quad \alpha \in \mathbb{C}, \quad n \in \ker \phi.$$

Taking the inner product of this decomposition with z, we get

$$\alpha = \frac{< z, x >}{\|z\|^2},$$

and evaluating ϕ on $x = \alpha z + n$, we find that

$$\phi(x) = \alpha\phi(z).$$

Eliminating α from these equations and rearranging, we obtain

$$\phi(x) =< x, y > .$$

where

$$y = \frac{\overline{\phi(z)}}{\|z\|^2} z.$$

Thus, every bounded linear functional can be represented as an inner product with a fixed vector. We have already seen that $\phi_y(x) =< x, y >$ defines a bounded linear functional on H for every $y \in H$. To prove that there is a unique y in H associated with a given linear functional, suppose that $\phi_{y_1} = \phi_{y_2}$. Then $\phi_{y_1}(y) = \phi_{y_2}(y)$ where $y = y_1 - y_2$, which implies $\|y_1 - y_2\|^2 = 0$, so $y_1 = y_2$. □

Theorem 1.1.6. *Let $T \in B(H)$. If $\|T\| < 1$, then the operator $I - T$ is invertible and*

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k. \quad \textit{(is called the } \textbf{\textit{Neumann series}}\textit{).}$$

Proof. See[17] Since $\|T\| < 1$ and for any $k \in \mathbb{N}$, we have $\|T^k\| \leq \|T\|^k$, the series $\sum \|T^k\|$ converges in $\mathbb{R}$. Given that $(B(H), \|\|_{B(H)})$is a Banach space (complete), this implies that the operator series $\sum T^k$ also converges in $B(H)$. Let

$$u = \sum_{k=0}^{\infty} T^k = \lim_{K} \sum_{k=0}^{K\to\infty} T^k.$$

Then, for any $K \geq 1$

$$(I - T)(\sum_{k=0}^{K} T^k) = (\sum_{k=0}^{K} T^k)(I - T) = I - T^{K+1}$$

Let $(I - T^{K+1})$ converge in $B(H)$ to the identity operator I.
We have $(I - T)U = U(I - T) = I$. Now, assume that S_0 is invertible operator in $B(H)$. For any operator $T \in B(H)$, we can write:

$$S_0 + T = S_0(I + S_0^{-1}T),$$

This shows that $S_0 + T$ is invertible if and only if $I + S_0^{-1}T$ is invertible. In particular, if $\|S_0^{-1}T\| < 1$ then $I + S_0^{-1}T$ is invertible, and hence $S_0 + T$ is invertible as well. Therefore, $B(S_0, \|S_0^{-1}\|^{-1})$ is contained in $B(H)$.

□

Theorem 1.1.7. *Let H be a $\mathbb{C}$-Hilbert space and let $T \in B(H)$, then:*

$$< Tx, x >= 0, \quad \forall x \in H \Longleftrightarrow T = 0$$

Proof. Clearly, if $T = 0$, then $(Tx, x) = 0$.
Let $x, y \in H$, then $x + y \in H$ and

$$< Tx + y, x + y >=< Tx, y > + < Ty, x >= 0...(1).$$

Replacing y by iy in (1), we obtain the following system:

$$\begin{cases} < Tx, y > + < Ty, x >= 0 \\ -i < Tx, y >) + i < Ty, x >= 0. \end{cases}$$

Solving the system, we get

$$< Tx, y >= 0 \quad \forall x, y \in H,$$

so it is true in particular $y = Tx$,

$$< Tx, Tx >= 0, \quad \forall x \in H,$$

therefore

$$\|Tx\| = 0, \quad \forall x \in H \Longleftrightarrow Tx = 0, \quad \forall x \in H.$$

Hence

$$T = 0$$

□

Remark 1.1.2. *The previous theorem is not true for $\mathbb{R}$-Hilbert.*
For example,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is bounded on $\mathbb{R}^2$ *and*

$$< A(x,y),(x,y) >= yx - xy = 0, \quad \forall (x,y) \in \mathbb{R}^2$$

but

$$A \neq 0$$

.

Corollary 1.1.1. *Let* $T \in \mathcal{B}(H)$ *and* $S \in \mathcal{B}(H)$. *We have*

$$\text{If } \forall x \in H, < Tx, x >=< Sx, x >, \text{ then } T = S$$

1.2 Adjoint in Hilbert spaces

See[15] Let H be a Hilbert space and $T : H \longrightarrow H$ be a linear transformation. We call the adjoint of T the function, denoted $T^* : H \longrightarrow H$ and defined by

$$< Tx, y >=< x, T^*y > \quad \forall x, y \in H.$$

Theorem 1.2.1. *Let* H *be a* $\mathbb{K}$*-Hilbert space. Let* $T : H \longrightarrow H$ *be a bounded linear operator. Then there exists a unique bounded operator, denoted* $T^* : H \longrightarrow H$.

Proof. Let $y \in H$ and φ be a function defined by:

$$\begin{aligned} \varphi : H \longrightarrow & \mathbb{K} \\ x \longmapsto & \varphi(x) =< Tx, y > . \end{aligned}$$

φ is obviously a **linear form** and it is bounded as

$$|\varphi(x)| = | < Tx, y > | \leq \|Tx\| \; \|y\| \leq \|T\| \; \|y\| \; \|x\|.$$

By Theorem 1.1.5, there exists a unique a in H such that

$$\varphi(x) =< Tx, y >=< x, a > .$$

Set $a = T^*y$, so

$$< Tx, y >=< x, T^*y > .$$

Clearly T^* is a linear transformation and it is bounded as

$$\|T^*y\| = \|a\| = \|\varphi\| = \sup_{\|x\|\leq 1} |(Tx, y)| = \sup_{\|x\|\leq 1} \|Tx\| \; \|y\|.$$

Hence

$$\|T^*y\| \leq \|T\| \; \|y\|.$$

□

Examples 1.2.1.

1. *Let H be a Hilbert space. The operator I (identity), its adjoint is $I^* = I$.*

2. *For the matrix Operator $M \in M_2(\mathbb{C}) = (a_{ij})$, its adjoint is $M^* = (\overline{a_{ij}})$, where $\overline{a_{ij}}$ is the complex conjugate.*

3. *Let $H = l^2\mathbb{C}$. The operator (shift) S, its adjoint S^* is defined by*

$$S^*(x_1, x_2, x_3, ...) = (x_2, x_3, x_4, ...)$$

4. *Let $H = \mathcal{C}([0,1], \mathbb{C})$. The operator $T : H \longrightarrow H$ defined by*

$$x \longmapsto T_\varphi f(x) = \varphi(x) f(x)$$

*is called the **multiplication operator**, its adjoint T^* is defined by*

$$T_\varphi^* = T_{\overline{\varphi}}$$

where $\overline{\varphi}(x)$ is the complex conjugate of $\varphi(x)$

Properties 1.2.1. *Let $T, S \in B(H)$. We have*

1. $\|T\| = \|T^*\|$.

2. $(T + S)^* = T^* + S^*$.

3. $(\alpha T)^* = \overline{\alpha} T^*$.

4. $(T^*)^* = T$.

5. $(ST)^* = T^* S^*$.

6. $\|TT^*\| = \|T^*T\| = \|T\|^2$.

Definition 1.2.1. *Let $T \in B(H)$. On a :*
$\ker T = \{x \in H : T(x) = 0\}$ *(is called the "**Kernel**" of T)*
$\text{Im } T = \{Tx, x \in H\}$

Proposition 1.2.1. *Soit $T \in B(H)$. On a:*

1. $\ker T^* = (\text{Im } T)^\perp$
2. $\ker T = (\text{Im } T^*)^\perp$

1.3 A Class of Bounded Operators

Definition 1.3.1. *Let $T \in B(H)$ where H is a $\mathbb{C}$-Hilbert space. We say that T is called :*

1. ***self-adjoint*** *if $T = T^*$.*
2. ***normal*** *if $TT^* = T^*T$.*
3. ***unitary*** *if $TT^* = T^*T = I$.*
4. *an **orthogonal projection** if T is self-adjoint and $T^2 = T$.*
5. *an **isometry** if $T^*T = I$.*
6. ***positive*** *if $(Tx, x) \geq 0$.*
7. ***invertible*** *if $S \in B(H)$, $TS = ST = I$. (We denote $S = T^{-1}$).*

Remark 1.3.1.

1. *positive on $\mathbb{R}$ this means that T is self-adjoint and $< Tx, x > \geq 0$*
2. *T self-adjoint $\Longrightarrow$ T normal*
3. *T unitary $\Longrightarrow$ T isometry, normal and invertible.*
4. *T positive $\Longrightarrow$ T self-adjoint.*

Property 1.3.1. *Let $T \in B(H)$. We have:*

1. *T is normal* $\Longleftrightarrow \|Tx\| = \|T^*x\|, \forall x \in H$

2. *T is self-adjoint* $\Longleftrightarrow < Tx, x > \in \mathbb{R}, \forall x \in H$ *on* $\mathbb{C}$.

3. *T is unitary* $\Longleftrightarrow \|Tx\| = \|T^*x\| = \|x\|, \forall x \in H$. *See[15]*

Proof.

1. We have:
$$\|Tx\|^2 - \|T^*x\|^2 =< (T^*T - TT^*)x, x > .$$
Hence
$$T \text{ normal} \Longleftrightarrow \|Tx\| = \|T^*x\|.$$

2. If T is self-adjoint (i.e $T = T^*$), then
$$< Tx, x >=< x, T^*x >=< x, Tx >= \overline{< Tx, x >}.$$
So
$$(Tx, x) \in \mathbb{R}, \forall x \in H.$$
Conversely, if $< Tx, x > \in \mathbb{R}$, then:
$$< Tx, x >= \overline{< Tx, x >} =< x, Tx >=< T^*x, x > \ \forall x \in H.$$
Hence T is self-adjoint.

3. If T is unitary (i.e $TT^* = T^*T = I$), then
$$\|Tx\|^2 =< Tx, Tx >=< x, T^*Tx >=< x, Ix >= \|x\|^2$$
and
$$\|T^*x\|^2 =< T^*x, T^*x >=< x, TT^*x >=< x, Ix >= \|x\|^2.$$
Conversely, if $\|Tx\| = \|T^*x\| = \|x\|$, then T is normal by property 1. Hence
$$\|Tx\|^2 =< Tx, Tx >=< x, x > .$$
Therefore, for all $x \in H$, we have
$$< Tx, Tx > - < x, x >=< T^*Tx, x > - < x, x >=< (T^*T - I)x, x >= 0$$

and

$$< T^*x, T^*x > - < x, x > = < TT^*x, x > - < x, x > = < (TT^* - I)x, x > = 0.$$

So

$$T^*T = TT^* = I.$$

Hence, T is unitary.

$\square$

Examples 1.3.1.

1. *The identity $I : H \longrightarrow H$ is self-adjoint and positive as*

$$< Ix, x > = \|x\|^2 \geq 0 \quad \forall x \in H.$$

2. *Let $S : l^2(\mathbb{C}) \longrightarrow l^2(\mathbb{C})$ be a shift operator, we recall that*

$$S(x_1, x_2, x_3, ..., x_n, ...) = (0, x_1, x_2, x_3, ..., x_n, ...).$$

We know that

$$S^*(x_1, x_2, x_3, ..., x_n, ...) = (x_2, x_3, x_4, ...).$$

S is not self-adjoint as

$$Se_1 \neq S^*e_1 \quad \textit{where} \quad e_1 = (1, 0, 0, 0, ...).$$

S is not positive as

$$(Sx, x) = -1 \quad \textit{where} \quad x = (-1, 1, 0, 0, ...).$$

on the other hand

$$S^*Sx = S^*(0, x_1, x_2, x_3, ..., x_n, ...) = (x_1, x_2, x_3, ..., x_n, ...) = I_{l^2}$$

and

$$SS^*x = S(x_2, x_3, x_4, ...) = (0, x_2, x_3, x_4, ...).$$

So S is not normal and it is not unitary, but it is an isometry.

3. *Let $H = L^2[0,1]$ and $T_\varphi \in B(H)$ such that*

$$T_\varphi f(x) = \varphi(x) f(x)$$

where φ is continuous on $[0,1]$ with complex value.
We know that $T_\varphi^* f(x) = \overline{\varphi(x)} f(x)$, that is
T_φ is self-adjoint $\Longleftrightarrow \varphi = \overline{\varphi}$.
T_φ is always normal.
T_φ is unitary $\Longleftrightarrow |\varphi| = 1$.
T_φ is positive $\Longleftrightarrow \varphi$ is positive.

Proposition 1.3.1. *Let $T \in B(H)$, we have*

$$T \text{ is invertible} \Longleftrightarrow \begin{cases} \exists \alpha > 0, \|Tx\| \geq \alpha \|x\|, \forall x \in H \\ \ker T^* = \{0\} \end{cases}$$

Example 1.3.1. *The shift operator S is not invertible as*

$$u = (1, 0, 0, 0, ...) \neq 0_{\ell^2} \quad \text{and} \quad S^*(u) = 0$$

i.e.

$$\ker S^* \neq \{0\}.$$

Remark 1.3.2. *If T is invertible, then T^* is invertible.*

Proposition 1.3.2. *If T is normal, then* $\ker T = \ker T^*$

Corollary 1.3.1. *Let $T \in B(H)$, we have:*

$$T \text{ is normal and invertible} \Longleftrightarrow \exists \alpha > 0, \forall x \in H, \|Tx\| \geq \alpha \|x\|$$

1.4 operator-valued maps

We consider functions with values in $B(H)$. More precisely, let J be an open interval in $\mathbb{R}$, and let us consider a map $F : J \to B(H)$.
The notion of continuity on $B(H)$ can be analyzed with respect to several topologies; however, in this work we will focus on only three of them.

Definition 1.4.1. *The map F is continuous in norm on J if for all $t \in J$*

$$\lim_{\varepsilon \to 0} \|F(t+\varepsilon) - F(t)\| = 0$$

The map F is strongly continuous on J if for any $f \in H$ and all $t \in J$

$$\lim_{\varepsilon \to 0} \|F(t+\varepsilon)f - F(t)f\| = 0$$

The map F is weakly continuous on J if for any $f, g \in H$ and all $t \in J$

$$\lim_{\varepsilon \to 0} < g, (F(t+\varepsilon) - F(t))f >= 0$$

One writes respectively $u - \lim_{\varepsilon \to 0} F(t+\varepsilon) = F(t)$, $s - \lim_{\varepsilon \to 0} F(t+\varepsilon) = F(t)$ and $w - \lim_{\varepsilon \to 0} F(t+\varepsilon) = F(t)$.

Definition 1.4.2. *The map F is differentiable in norm on J if there exists a map $F' : J \to B(H)$ such that*

$$\lim_{\varepsilon \to 0} \|\frac{1}{\varepsilon}(F(t+\varepsilon) - F(t)) - F'(t)\| = 0$$

The concepts of strong differentiability and weak differentiability share similar definitions If J is an open interval of $\mathbb{R}$ and if $F : J \to B(H)$, one defines $\int_J F(t)dt$ as a Riemann integral (limit of finite sums over a partition of J) if this limiting procedure exists and is independent of the partitions of J.Note that these integrals can be defined in the weak topology, in the strong topology or in the norm topology. For example, if $F : J \to B(H)$ is strongly continuous and if $\int_J \|F(t)\|dt < \infty$, then the integral $\int_J F(t)dt$ exists in the strong topology. See[16]

Chapter 2

Unbounded Operators

In the basic theory of Hilbert and Banach spaces, the linear operators typically considered are those acting within such spaces or between them. However, the study of differential operators leads to the consideration of unbounded linear operators, due to their importance. Here $T : H \longrightarrow H$ need not be defined on the entire space H but may instead be defined on a linear subset, $D(T)$, which is called the domain of T. Among these domains, there is the **maximal domain** which is defined by $D(T) = \{x \in H : Tx \in H\}$. Of Particular importance are operators whose domain is dense in H (i.e., with $\overline{D(T)} = H$). When T is bounded and densely defined, it admits a unique continuous extension to a bounded operator on all of H, and thus belongs to the space $B(H)$. However, if T is unbounded, such an extension generally does not exist. For operators of this type, another significant property to consider is whether the operator is closed.
Before introducing further concepts, here are some examples of unbounded operators.

Examples 2.0.1.

1. *Let T be an operator defined by*

$$\begin{aligned} T : L^2(\mathbb{R}) &\longrightarrow L^2(\mathbb{R}) \\ f &\longmapsto Tf(x) = (1+x^2)f(x). \end{aligned}$$

Let's show that T is unbounded on $D(T) = \{f \in L^2(\mathbb{R}) : (1+x^2)f \in L^2(\mathbb{R}), \}$. Let $(f_n)_{n\geq 1}$ a sequence defined by

$$f_n(x) = \frac{e^{-\frac{x^2}{2n^2}}}{1+x^2} \ , \ \ n \in \mathbb{N}^*.$$

We have $(f_n)_{n\geq 0} \subset L^2(\mathbb{R})$ *as*

$$0 \leq f_n(x) \leq \frac{1}{\sqrt{1+x^2}} \quad and \quad \int_{-\infty}^{+\infty} \frac{1}{1+x^2}dx = \pi < +\infty.$$

Therefore

$$\|f_n\| \leq (\int_{-\infty}^{+\infty} \frac{1}{1+x^2}dx)^{\frac{1}{2}} = \sqrt{\pi}.$$

i.e. the sequence $(f_n)_{n\geq 1}$ *is bounded.*
Using the ***Gaussian integral*** $\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$, *we obtain*

$$\|Tf_n\| = (\int_{-\infty}^{+\infty} ((1+x^2) \times \frac{e^{-\frac{x^2}{2n^2}}}{1+x^2})^2 dx)^{\frac{1}{2}} = \pi^{\frac{1}{4}}\sqrt{n}.$$

Since $\lim\limits_{n\longrightarrow\infty} \pi^{\frac{1}{4}}\sqrt{n} = +\infty$, *the sequence* $(Tf_n)_{n\geq 1}$ *is not bounded. Hence* T *is unbounded.*

2. *The operator*

$$\begin{aligned} T : \ell^2(\mathbb{N}) &\longrightarrow \ell^2(\mathbb{N}) \\ (x_n) &\longmapsto T(x_n) = (nx_n) \end{aligned}$$

is unbounded on its maximal domain $D(T)$ *as*

$$\|Te_n\| = n\|e_n\| \quad and \quad n \longrightarrow +\infty$$

where $e_n = (0, 0, 0, ..., \underbrace{1}_{n\text{-}th}, 0, 0, ...)$.

2.1 Closed operators

Definition 2.1.1. *Let* H *be a Hilbert space and let* S *and* T *be two linear operators from* H *into* H. *We shall say that* T *is an extension of* S *or that* S *is a restriction of* T *and write* $S \subseteq T$, *or equivalently* $T \supseteq S$, *when* $D(S) \subseteq D(T)$ *and* $S(x) = T(x)$ *for all* $x \in D(S)$. *That is, we have* $S \subseteq T$ *if and only if* $S = T \mid D(S)$. *If we have two operators* T *and* S *defined on their domain* $D(T)$ *and* $D(S)$ *respectively, then the domain of the* ***sum*** $T + S$ *is* $D(T + S) = D(T) \cap D(S)$. *Also, the domain of the* ***product*** ST *is* $D(ST) = \{x \in D(T) : Tx \in D(S)\}$. *See[7]*

Proposition 2.1.1. *Let T, S and R be unbounded operators defined on $D(T)$, $D(S)$ and $D(R)$ respectively. Then $(T+S)R = TR + SR$ and $R(T+S) \supseteq RT + RS$.*

Definition 2.1.2. *Let $T : D(T) \longrightarrow H$ be an unbounded operator. T is said to be* ***closed*** *when the graph of T, denoted and defined by*

$$G(T) = \{(x, Tx) : x \in D(T)\}$$

is a ***closed*** *subspace in $H \times H$ or $H \oplus H$ and we say T is* ***closable*** *if there exists a closed linear operator $S : H \longrightarrow H$ such that $T \subseteq S$. The closure of an operator T, denoted $\overline{T}$, is defined as the minimal closed operator that extends T. It serves as the* ***smallest closed*** *operator containing all the information of T while ensuring closedness . See[15]*

Remark 2.1.1. *Let $< .,. >$ and $\|.\|$ denote the scalar product and norm on H. It is easily seen that*

$$< x, y >_T = < x, y > + < Tx, Ty >, \qquad x, y \in D(T)$$

defines a scalar product on $D(T)$. The corresponding norm

$$\|x\|_T = \|x\| + \|Tx\|, \qquad x \in D(T)$$

is called the ***graph norm*** *of T.*

Proposition 2.1.2. *Let T be an unbounded operator defined on $D(T) \subset H$. The following properties are equivalent.*

1. *T is closed.*

2. *$\forall (x_n)_n \subset D(A)$ such that $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$, then $x \in D(T)$ and $y = Tx$.*

3. *$(D(T), \|.\|_T)$ is a complete space.*

Proof. Clearly, (1) $\iff$ (2). Let's show that (1) $\implies$ (3). Let $(x_n)_{n\geq 0}$ be a Cauchy sequence in $(D(T), \|.\|_T)$. We have

$$\|x_n - x_m\|_T = \|x_n - x_m\|_H + \|Tx_n - Tx_m\|_H \longrightarrow 0,$$

therefore

$$\|x_n - x_m\|_H \longrightarrow 0 \quad \text{and} \quad \|Tx_n - Tx_m\|_H \longrightarrow 0$$

i.e. the sequences $(x_n)_{n\geq 0}$ and $(Tx_n)_{n\geq 0}$ are Cauchy in $(H, \|.\|_H)$ which is complete, so $x_n \longrightarrow x \in H$ and since T is closed, $x \in D(T)$, hence $(D(T), \|.\|_T)$ is complete.
Now, let $(x_n)_{n\geq 0} \subset D(T)$ be a sequence such that

$$x_n \longrightarrow x \quad and \quad Tx_n \longrightarrow y,$$

then these sequences are Cauchy in H, i.e.

$$\|x_n - x_m\|_T = \|x_n - x_m\|_H + \|Tx_n - Tx_m\|_H \longrightarrow 0,$$

So $(x_n)_{n\geq 0}$ is Cauchy in $(D(T), \|.\|_T)$, it follows that (x_n) converges to $x \in D(T)$, therefore

$$\|x_n - x\|_T = \|x_n - x\|_H + \|Tx_n - Tx\|_H \longrightarrow 0$$

it follows

$$\|Tx_n - Tx\|_H \longrightarrow 0,$$

hence

$$\lim_{n\longrightarrow\infty} Tx_n = Tx = y \quad \text{(the limit is unique).}$$

i.e.

$$x \in D(T) \quad and \quad y = Tx.$$

Whence T is closed. □

Definition 2.1.3. *Let T be an unbounded operator and let S be a bounded operator. We say that T and S commute if $ST \subset TS$.*

Definition 2.1.4. *Let T be an unbounded operator. We say that T is **invertible** if there exists a bounded operator S (i.e. $B \in B(H)$) such that* $\begin{cases} TS = I \\ ST \subset I \end{cases}$
*S is called the **inverse** of T, we denote $S = T^{-1}$ and it is unique.*

Example 2.1.1. *Let*

$$\begin{aligned} T : L^2(\mathbb{R}) \longrightarrow & L^2(\mathbb{R}) \\ f \longmapsto & Tf(x) = (x^2+1)f(x). \end{aligned}$$

T is defined on the maximal domain maximal $D(T) = \{f \in L^2(\mathbb{R}) : (x^2+1)f \in L^2(\mathbb{R})\}$. Let S be an operator such that

$$Sf(x) = \frac{1}{x^2+1} f(x).$$

S is bounded and $D(S) = L^2(\mathbb{R})$ with,

$$TSf(x) = T[\frac{1}{x^2+1} f(x)] = (x^2+1) \times \frac{1}{x^2+1} f(x) = f(x)$$

and

$$\begin{aligned}
D(TS) =& \{f \in D(S) : Sf \in D(T)\} \\
D(TS) =& \{f \in L^2(\mathbb{R}) : \frac{1}{x^2+1} f \in D(T)\} \\
D(TS) =& \{f \in L^2(\mathbb{R}) : \frac{1}{x^2+1} f \in L^2(\mathbb{R}) \text{ and } \frac{1}{x^2+1}(x^2+1)f \in L^2(\mathbb{R})\} \\
D(TS) =& \{f \in L^2(\mathbb{R}\} = L^2(\mathbb{R}).
\end{aligned}$$

Clearly, $STf = f$ and,

$$\begin{aligned}
D(ST) =& \{f \in D(T) : Tf \in D(S)\} \\
D(ST) =& \{f \in L^2(\mathbb{R})(x^2+1)f \in L^2(\mathbb{R}) : (1+x^2)f \in L^2(\mathbb{R})\} \\
D(ST) =& \{f \in L^2(\mathbb{R}) : (x^2+1)f \in L^2(\mathbb{R})\} \\
D(ST) =& D(T) \subsetneqq L^2(\mathbb{R}).
\end{aligned}$$

Hence T is invertible with $T^{-1} = S$.

Theorem 2.1.1. *Let T be an unbounded operator. If T is invertible, then T is closed.*

Proof. Let $D(T)$ be a domain of T. Since T is invertible, there exists $T^{-1} \in B(H)$ such that $TT^{-1} = I_H$ and $T^{-1}T = I_{D(T)}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D(T)$ such that

$$x_n \longrightarrow x \quad \text{and} \quad Tx_n \longrightarrow y.$$

Therefore

$$x \longleftarrow x_n = T^{-1}Tx_n \longrightarrow T^{-1}y \quad (\text{as } T^{-1} \text{ is bounded}).$$

Hence

$$x = T^{-1}y \in D(T) \quad \text{and} \quad Tx = TT^{-1}y = y.$$

□

2.2 Products and Sums of Closed Operators

In general, the product and sum of two closed operators are not necessarily closed.

Example 2.2.1. *Let T be a closed operator defined on the domain $D(T)$, then $0T$ is not closed as*

$$D(0T) = D(T) \quad \textit{and} \quad (0T)x = 0, \forall x \in D(T).$$

For $S = -T$ stays closed, then

$$T - T = 0_{D(T)}$$

is not closed $D(T)$.

Theorem 2.2.1. *Let S and T be closed operators with dense domains $D(A)$ and $D(B)$, respectively, then*

1. *ST is closed if* $\begin{cases} S & \textit{is invertible} \\ \textit{or} & \\ T & \textit{is bounded} \end{cases}$
2. *$T + S$ closed if S is bounded (i.e $S \in \mathcal{B}(H)$).*

Proof.

1. Suppose that S is invertible (i.e $S^{-1} \in \mathcal{B}(H)$).
   Let $(x_n)_{\geq 0}$ be a sequence in $D(ST)$ such that:

   $$x_n \longrightarrow x \quad \text{and} \quad STx_n \longrightarrow y,$$

   therefore

   $$\begin{aligned} STx \longrightarrow y \Longrightarrow & S^{-1}STx_n \longrightarrow S^{-1}y \quad (\text{as } S^{-1} \text{ is bounded}) \\ \Longrightarrow & Tx_n \longrightarrow S^{-1}y \quad (\text{as } S^{-1}S \subset I). \end{aligned}$$

   Since T is closed,

   $$x \in D(T) \;\; \textit{and} \;\; S^{-1}y = Tx \in D(S) \;\; (\text{since } S^{-1} : D(S) \longrightarrow \text{Im } S)$$

   so

   $$\begin{cases} Tx \in D(S) \\ x \in D(ST) \\ SS^{-1}y = y = STx \end{cases}$$

Hence, ST is closed.

Now, suppose that T is closed.

Let $(x_n)_{\geq 0}$ be a sequence in $D(ST)$ such that:

$$x_n \longrightarrow x \quad \text{and} \quad STx_n \longrightarrow y.$$

since T is closed,

$$Tx_n \longrightarrow Tx \quad \text{and} \quad x \in D(T).$$

Set $z_n = Tx_n$ and $z = Tx$ and by the closedness of the operator S, we obtain

$$Sz_n \longrightarrow Sz,$$

i.e,

$$Tx \in D(S) \ \ \textit{and} \ \ y = STx.$$

Hence ST is closed.

2. Since S is bounded,

$$D(T+S) = D(T).$$

Let $(x_n)_{n\geq 0}$ be a sequence in $D(T+S)$ such that:

$$x_n \longrightarrow x \quad \text{and} \quad (S+T)x_n \longrightarrow y.$$

Since S is bounded,

$$Sx_n \longrightarrow Sx,$$

it follows

$$y \longleftarrow Sx_n + Tx_n \longrightarrow Sx + Tx$$
$$Tx_n + Sx \longrightarrow y$$
$$Tx_n \longrightarrow y - Sx.$$

But T is closed, so

$$y - Sx = Tx \quad \text{and} \quad x \in D(T).$$

we deduce,

$$(S+T)x = y \ \ \textit{and} \ \ x \in D(T)$$

Hence $T+S$ is closed.

□

2.3 Adjoint of Unbounded Operators

Definition 2.3.1. *Let H be a Hilbert space and let $T : D(T) \longrightarrow H$ be an unbounded operator and $D(T)$ its domain. The domain of the **adjoint** of T is denoted and defined by*

$$D(T^*) = \{y \in H : \varphi(x) =< Tx, y > \textit{ is continuous on } D(T)\}$$

.

Remark 2.3.1. *If $y \in D(T^*)$ is fixed, then by [Hahn-Banach] Theorem, there exists a **bounded linear form** f on H such that $f \mid D(T) = \varphi =< T., y >$. Therefore*

$$\exists T^*y \in H :< Tx, y >=< x, T^*y >$$

*and it is very important to suppose that $D(T)$ is **dense** in H, as if $\overline{D(T)} \neq H$ and if $y_0 \in D(T)^{\perp} \neq \{0\}$, then:*

$$\forall x \in D(T) :< x, T^*y + y_0 >=< x, T^*y > + \underbrace{< x, y_0 >}_{=0} =< x, T^*y >,$$

In this case, we get another adjoint of T which is $T^*y + y_0$ (i.e the adjoint is not unique).

Lemma 2.3.1. *Let H be a Hilbert space. The function V is defined by*

$$V : H \times H \longrightarrow H \times H$$
$$(x, y) \longmapsto V(x, y) = (-y, x)$$

such that $H \times H$ is equipped with its usual scalar product. Then V is a unitary linear operator with $V^2 = -I$. Then for each M subset of $H \times H$ we have

$$V(M^{\perp}) = (V(M))^{\perp}$$

Theorem 2.3.1. *Let T, S and ST be unbounded operators with dense domains $D(T)$, $D(S)$ and $D(ST)$ respectively. Then*

1. *$T \subset S \Longrightarrow S^* \subset T^*$.*

2. *T^* is closed.*

3. *$(T + S)^* \supset T^* + S^*$.*

4. $(ST)^* \supset T^*S^*$.

5. $(T+S)^* = T^* + S^*$ *if* S *is bounded for example.*

6. $(ST)^* = T^*S^*$ *if* S *is bounded (it's not* T*)).*

Proof. Let us first show that for an unbounded operator T that

$$G_{T^*} = (V(G_T))^{\perp}$$

such that V is the function cited in Lemma 2.3.1.
Let $x \in D(T)$ and $y \in D(T^*)$. We have:

$$< (y, T^*y), V(x, Tx) >=< (y, T^*y), (-Tx, x) >= - < y, Tx > + < T^*y, x >= 0$$

Then

$$G_{T^*} \subseteq (V(G_T))^{\perp}$$

Conversely, if $(y, z) \in (V(G_T))^{\perp}$, then

$$< (y, z), (-Tx, x) >=< y, -Tx) > + < z, x >= 0, \forall x \in D(T).$$

So

$$< Tx, y >=< x, z >$$

We get,

$$y \in D(T^*) \quad \text{and} \quad z = T^*y.$$

i.e.,

$$(V(G_T))^{\perp} \subseteq G_{T^*}$$

Hence,

$$G_{T^*} = (V(G_T))^{\perp}$$

1. We have

$$\begin{aligned} T \subset S \Longrightarrow & G_T \subset G_S \\ \Longrightarrow & V(G_T) \subset V(G_S) \\ \Longrightarrow & (V(G_S))^{\perp} \subset (V(G_T))^{\perp} \\ \Longrightarrow & G_{S^*} \subset G_{T^*} \\ \Longrightarrow & S^* \subset T^* \end{aligned}$$

2. Since $M^\perp$ is always closed, then:

$$G_{T^*} = (V(G_T))^\perp \quad \text{is closed}$$

   Hence, A^* is closed.

3. We have:

$$D(T+S) = D(T) \cap D(S) \quad \text{and} \quad D(T^*+S^*) = D(T^*) \cap D(S^*).$$

   So for

$$x \in D(T+S) \quad \text{and} \quad y \in D(T^*+S^*),$$

   we obtain:

$$\begin{aligned} < x, (T^*+S^*)y > &= < x, T^*y > + < x, S^*y > \\ &= < Tx, y > + < Sx, y > \\ &= < (T+S)x, y > . \end{aligned}$$

   i.e.,

$$(T+S)^* \supset T^* + S^*$$

   .

4. Let $y \in D(T^*S^*)$ and $x \in D(ST)$,
   i.e.,

$$y \in D(S^*) \quad \text{and} \quad S^*y \in D(T^*) \quad \text{and} \quad x \in D(T) \quad \text{and} \quad Tx \in D(S).$$

   Therefore,

$$< Tx, S^*y >=< x, T^*S^*y > \quad (\text{as } x \in D(T) \text{ and } S^*y \in D(T^*)),$$

   and on the other hand,

$$< STx, y >=< Tx, S^*y > \quad (\text{as } Tx \in D(S) \text{ and } y \in D(S^*)),$$

   it follows,

$$< STx, y >=< x, T^*S^*y >$$

5. We have B is bounded, then:

$$D(T+S) = D(T) \quad \text{and} \quad D(T^* + S^*) = D(T^*).$$

Let $y \in D((T+S)^*)$, then for each $x \in D(T)$, we obtain

$$< Tx, y >=< (T+S)x, y > - < Tx, y >=< x, (T+S)^* y > - < x, S^* y >$$

Hence,

$$y \in D(T^*) = D(T^* + S^*).$$

6. Clearly, $(ST)^* \supset T^* S^*$. Let us show that $(ST)^* \subset T^* S^*$.
If S is bounded, then:

$$D(S) = D(S^*) = H.$$

Let $y \in D((ST)^*)$ and $x \in D(ST) = D(T)$, therefore:

$$< Tx, S^* y >=< STx, y >=< x, (ST)^* y >, \forall x \in D(ST),$$

It follows

$$S^* y \in D(T^*),$$

so

$$y \in D(T^* S^*),$$

hence

$$(ST)^* \subset T^* S^*$$

□

The following proposition is very important.

Proposition 2.3.1. *Let T be an unbounded operator. We have T is closeable if T^* has a dense domain, and in this case, we have*

$$(T^*)^* = T^{**} = \overline{T}$$

Lemma 2.3.2. *Let T be an unbounded operator with a dense domain.*
If T is invertible, then:

$$(T^*)^{-1} = (T^{-1})^*$$

Proposition 2.3.2. *Let T and S be two unbounded operators with dense domains. If T is invertible, then*

$$(ST)^* = T^*S^*$$

Proof. We always have $(TS)^* \supset T^*S^*$ (by Theorem 2.3.1).
Let's show that $(ST)^* \subset T^*S^*$.
We have:

$$\begin{aligned} STT^{-1} = S \Longrightarrow & (T^{-1})^*(ST)^* \subset (STT^{-1})^* = S^* \\ \Longrightarrow & (T^*)^{-1}(ST)^* \subset S^* \text{ (by Lemma2.3.2)} \\ \Longrightarrow & \underbrace{T^*(T^*)^{-1}}_{=I}(ST)^* \subset T^*S^* \\ \Longrightarrow & (ST)^* \subset T^*S^*. \end{aligned}$$

Hence,

$$(ST)^* = T^*S^*.$$

□

2.4 Symmetric, Self-Adjoint and Normal Operators

In this section, we see some classes of unbounded operators.

Definitions 2.4.1. *Let T be an unbounded operator with a dense domain $D(T)$. We say that*

1. *T is symmetric if $T \subset T^*$.*
2. *T is self-adjoint if $T = T^*$.*
3. *T is essentially self-adjoint if $\overline{T} = (\overline{T})^*$.*
4. *T is normal if T is closed and $TT^* = T^*T$*

Lemma 2.4.1. *Let T be a closed operator with a dense domain $D(T)$. We have:*
If T' is a restriction of T on $D(T^*T)$, then $G_{T'}$ is dense in G_T (i.e. $\overline{G_{T'}} = G_T$).

Theorem 2.4.1. *If N is a normal operator. Then*

1. $D(N) = D(N^*)$
2. $\|Nx\| = \|N^*x\| \; \forall x \in D(N)$

Proof.

1. If $y \in D(NN^*) = D(N^*N)$, then

   $$< Ny, Ny >=< y, N^*Ny > \quad \text{as } Ny \in D(N^*)$$

   and

   $$< N^*y, N^*y >=< y, NN^*y > \quad \text{as } N^*y \in D(N).$$

   Hence,

   $$\|Ny\| = \|N^*y\| \quad \text{and } NN^*y = N^*Ny.$$

   Let $x \in D(N)$. If N' is a restriction of N on $D(N^*N)$, then by Lemma 2.4.1, there exists a sequence $(x_n)_{n\geq 0}$ in $D(N^*N)$ such that

   $$\|x_n - x\| \longrightarrow 0 \quad \text{for} \quad n \longrightarrow +\infty$$

   and

   $$\|Nx_n - Nx\| \longrightarrow 0 \quad \text{for} \quad n \longrightarrow +\infty.$$

   Since $\|N^*x_p - N^*x_q\| = \|Nx_p - Nx_q\|$, we obtain $(N^*x_n)_{n\geq 0}$ which is a Cauchy in H. Therefore, there exists z in H such that

   $$\|N^*x_n - z\| \longrightarrow 0 \quad \text{for} \quad n \longrightarrow +\infty$$

   And since N^* is closed,

   $$x \in D(N^*) \quad \text{and} \quad z = N^*x.$$

   Hence

   $$D(N) \subset D(N^*).$$

   Similarly, we get

   $$D(N^*) \subset D(N) \quad \text{as} \quad N^{**} = N.$$

   Hence,

   $$D(N) = D(N^*)$$

2. We have
$$\|N^*x\| = \|z\| = \lim_{n\longrightarrow\infty} \|N^*x_n\| = \lim_{n\longrightarrow\infty} \|Nx_n\| = \|Nx\|$$

□

Remark 2.4.1.

1. *Every symmetric operator is closable.*

2. *Every self-adjoint operator is closed.*

3. *Every self-adjoint operator is normal.*

4. *Every normal and symmetric operator is self-adjoint.*

The following theorem is widely used.

Theorem 2.4.2. *Let T be a closed unbounded operator with a dense domain. Then the operator T^*T is self-adjoint on $D(T^*T)$.*

2.5 Fuglede-Putnam Theorem

The following is the Fuglede-Putnam theorem, it is most important in the next chapter. See[2]

Theorem 2.5.1 (Bounded case)**.** *Let T, M and $N \in B(H)$. If N and M are normal and $TN = MT$, then*
$$TN^* = M^*T$$

Theorem 2.5.2 (Unbounded case)**.** *Let $T \in B(H)$ and let N and M be unbounded operators. If N and M are normal and $TN \subseteq MT$, then*

$$TN^* \subseteq M^*T$$

Chapter 3

Normality and Maximality of Operators in Hilbert Space

One of the main aims of this chapter is to seek conditions that transform $S \subset T$ into $S = T$ (which we call a maximality condition) for some classes of operators and also in the case of a product of two operators. This type of result is a powerful tool when proving results on unbounded operators. For example, Statement (3) of the following theorem plays a role in the proof of "the unbounded" version of the spectral theorem for normal operators. For other uses, let us now list some known maximality results:

3.1 Maximality of Linear Operators

Theorem 3.1.1. *See[3] Let S, T be two operators with (dense when necessary) domains $D(S)$ and $D(T)$ respectively, such that $S \subset T$. Then $S = T$ if any of the following conditions is satisfied:*

1. *S is surjective, and T is injective.*
2. *T is symmetric, and S is self-adjoint (resp. normal). We then say that self-adjoint (resp. normal) operators are maximally symmetric.*
3. *T and S are normal (we say that normal operators are maximally normal). Hence, self-adjoint (resp. normal) operators are maximally normal (resp. self-adjoint).*

Proof. Let's show properties 1 and 2.

1. Let $x \in D(T)$. Since S is surjective, there is a $y \in D(S)$ such that $Tx = Sy$. From $S \subset T$ we get $Tx = Ty$, so $x = y$, because T is injective. Hence, we have $x = y \in D(S)$. Thus, $D(T) \subseteq D(S)$, whence $S = T$.

2. We have $S \subset T \Longrightarrow T^* \subset S^*$. Since T and S are normal, $D(T) = D(T^*)$ and $D(S) = D(S^*)$. It follows

$$D(S) \subset D(T) = D(T^*) \subset D(S^*) = D(S),$$

   thus $D(S) = D(T)$ and hence $S = T$.

□

Proposition 3.1.1. *Let S be a densely defined operator such that $S \subset T$ and $S \subset T^*$. If $D(T) = D(T^*)$, then T is self-adjoint.*

Proof. For all $x \in D(T) = D(T^*)$ and for all $y \in D(S) \subset D(T) = D(T^*)$ we may write

$$\begin{aligned} < Tx, y > &=< x, T^*y > \\ &=< x, Sy > \\ &=< x, Ty > \\ &=< T^*x, y > . \end{aligned}$$

Since D(S) is dense, it follows that $Tx = T^*x$ for all $x \in D(T) = D(T*)$, that is, T is self-adjoint. □

The following theorem gives the equality for the product of two operators.

Theorem 3.1.2. *Let R, S and T be operators such that $T \subset RS$. If R, S and T are self-adjoint, then $T = RS$.*

3.2 Normality of Products and Sums of Operators

Now we see some theorems on the normality of products of normal bounded operators. not necessarily bounded.

Theorem 3.2.1. *Let T and S be normal operators such that $T, S \in B(H)$. If $TS = ST$, then TS and ST are normal.*

Proof. Since $TS = ST$, Fuglede-Putnam theorem yields

$$TS^* = S^*T.$$

and by adjoint, it follows

$$ST^* = T^*S.$$

Therefore

$$\begin{aligned}(TS)^*TS &= S^*T^*TS \\ &= S^*TT^*S \\ &= TS^*T^*S \\ &= TT^*S^*S.\end{aligned}$$

We also have

$$\begin{aligned}TS(TS)^* &= TSS^*T^* \\ &= TST^*S^* \\ &= TT^*SS^*\end{aligned}$$

Since T and S are normal,

$$(TS)^*TS = TS(TS)^* = TT^*SS^*,$$

and this marks the normality of TS and ST. □

The following proposition gives the self-adjointness of the product of two self-adjoint operators.

Proposition 3.2.1. *Let T and S be self-adjoint operators such that $T, S \in B(H)$. If $TS = ST$, then TS and ST are self-adjoint.*

Remark 3.2.1. *In the previous theorem and proposition, commutativity is necessary because, in general, if S and T are normal, this does not imply that ST or TS are normal.*

Example 3.2.1. *In $\mathbb{R}^2$, consider the following self-adjoint matrices*

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

We can easily check that $TS \neq TS$ and they are not self-adjoint (so they are not normal).

Theorem 3.2.2. *Let T and S be normal operators in $B(H)$. If $TS = ST$, then TS and $S + T$ are normal.*

Proof. see[15] Since $TS = ST$, the Fuglede-Putnam theorem gives us:

$$TS^* = S^*T...(1)$$

and by adjoint, we obtain

$$ST^* = T^*S...(2).$$

Indeed

$$\begin{aligned}(T + S)^*(T + S) &= (T^* + S^*)(T + S) \\ &= T^*T + T^*S + S^*T + S^*S\end{aligned}$$

and

$$\begin{aligned}(T + S)(T + S)^* &= (T + S)(T^* + S^*) \\ &= TT^* + TS^* + ST^* + SS^*\end{aligned}$$

□

Combining with the equalities (1) and (2) and since T and S are normal, we get

$$(T + S)^*(T + S) = T + S)(T + S)^* = T^*T + T^*S + S^*T + S^*S$$

Hence $T + S$ is normal.
It's important to note that the sum of two normal operators is not necessarily normal, as shown in

Example 3.2.2. *Consider for instance in $\mathbb{R}^2$ the matrices T and S defined by*

$$T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We can check that T and S are normal, but

$$T + S = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$$

is not normal, i.e. the condition of commutativity between T and S is very important.

Now we see some theorems on the normality of products of normal operators not necessarily bounded.

Theorem 3.2.3. *Let T, S be unbounded operators with $S^{-1} \in B(H)$. If $TS \subset ST$, then ST and $\overline{TS}$ are both normal when TS is densely defined.*

Proof. Since S is invertible, we begin with the relation.

$$\begin{aligned} TS \subset ST &\Longrightarrow T \subset STS^{-1} \\ &\Longrightarrow S^{-1}T \subset TS^{-1}. \end{aligned}$$

Applying the Fuglede theorem, we have

$$S^{-1}T^* \subset T^*S^{-1}.$$

Left multiplying, then right multiplying by S yields

$$T^*S \subset ST^*$$

or

$$TS^* \subset S^*T.$$

Hence,

$$\begin{aligned} (ST)^*ST &\subset (TS)^*ST \\ &= S^*T^*ST \\ &\subset S^*ST^*T \\ &= S^*ST^*T. \end{aligned}$$

But ST is closed, so $(ST)^*ST$ is self-adjoint. Since S^*S and T^*T are also self-adjoint, Theorem 3.1.2 gives us

$$(ST)^*ST = S^*ST^*T.$$

Similarly, we obtain

$$ST(ST)^* = SS^*TT^*.$$

Since T and S are normal, we obtain

$$(ST)^*ST = ST(ST)^*,$$

establishing the normality of ST.

Now, let's prove that $\overline{TS}$ is normal. Observe that

$$ST \subset \lambda ST \Longrightarrow T^*S^* \subset \lambda S^*T^*.$$

Also, since S is invertible, B^* is also invertible. Thus the first part of the proof, S^*T^* is normal and so is $(TS)^* = S^*T^*$. Hence its adjoint $(TS)^{**} = \overline{TS}$ stays normal. □

Theorem 3.2.4. *Let T and S be normal operators with $S \in B(H)$. If $ST \subset TS$, then TS and $\overline{ST}$ are both normal (and so $TS = \overline{ST}$).*

Proof. Since $ST \subset TS$, Fuglede Theorem yields $ST^* \subset T^*S$. Hence (since also TS is densely defined).

$$\begin{aligned}(TS)^*TS &\supset S^*T^*TS \\ &= S^*TT^*S \\ &\supset S^*TSS^* \\ &\supset S^*STT^*.\end{aligned}$$

Since TS is closed, it follows that $(TS)^*TS$ is self-adjoint, and by the boundedness of S^*S, we get

$$(TS)^*TS \subset TT^*S^*S$$

or merely

$$(TS)^*TS = TT^*S^*S = TT^*SS^*$$

by Theorem 3.1.2. Similarly, we obtain

$$TS(TS)^* = TT^*SS^*,$$

and this marks the end of the proof of the normality of TS. To show that $\overline{ST}$ is normal, we first observe that

$$\overline{ST} = (ST)^{**} = (T^*S^*)^*.$$

Now, since $ST \subset TS$, clearly

$$S^*T^* \subset T^*S^*.$$

The first part of the proof leads to the normality of T^*S^* because both T^* and S^* are normal. Accordingly, $(T^*S^*)^*$ too is normal, that is, $\overline{ST}$ is normal. □

Corollary 3.2.1. *Let T and S be self-adjoint operators with $S \in B(H)$. If $ST \subset TS$, then TS and $\overline{ST}$ are both self-adjoint.*

Proof. Since $ST \subset TS$, and T and S are self-adjoint, the previous theorem yields the normality of $\overline{ST}$. But

$$ST \subset TS \Longrightarrow \overline{ST} \subset TS = (ST)^*,$$

which means that $\overline{ST}$ is also symmetric. Therefore, $\overline{ST}$ is self-adjoint. Consequently,

$$TS = (ST)^* = \overline{ST},$$

and so TS is also self-adjoint, completing the proof. □

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