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Dedications

I dedicate this work: To my mother, for her love, encouragement, and countless sacrifices. To my father, for his unshakable support, affection, and instrumental faith in me. To the memory of my brother. To all the members of my family. To all my dear friends. And to everyone who has loved me and believed in me

lilia

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ABSTRACT

Fractional calculus is a generalization of classical calculus that extends the concept of derivatives and integrals to non-integer orders. In the first chapter, we studied the Riemann–Liouville and Hilfer fractional derivatives, as well as fractional integrals and their properties , and in Chapter 2 We consider a basic fractional differential inequality involving a fractional derivative known as the Hilfer derivative and a polynomial source term. A nonexistence result for global solutions is proved in a suitable functional space, and in chapter 3 ,Mohammed D. Kassim and Thabet Abdeljawad analyze a system of nonlinear fractional differential equations involving two types of fractional derivatives: the Caputo derivative and the Riemann–Liouville derivative. The source terms are nonlocal in time, making the system more general than those usually studied. The authors establish nonexistence results by using the test function method, specific properties of fractional derivatives, and integral inequalities .

keys words and phrases : non Existence result, fractional system , fractional differential equation, Riemann-Liouville fractional derivative, gobal solution , integral fractionnaire de Riemann-Liouville

CONTENTS

0.1 General Introduction

Integer-order derivatives and integrals generally have clear physical and geo-metric interpretations see [13], [20] . This is why their use in solving applied problems in various scientific fields is straight forward. The absence of an answer to this question has made the theory of fractional differentiation and integration quite mysterious. Consequently, it has remained one of the open problems in the field. Fractional differentiation and integration are generalizations of classical integer-order differentiation and integration. For this reason, it would be very interesting to have physical and geometric interpretations of fractional-order opera-tors, which would provide a link to the classical interpretations of integer-order calculus. The physical interpretation of fractional integration and differentiation relies on the use of two types of time : cosmic time and individual time, , whereas classical differential and integral calculus is based on the use of mathematical time

Recently, several papers have been published on the study of differential equations involving the derivative of Hilfer. . The applications of fractional derivatives of this type are numerous. However, relatively little has been published on the Hilfer-type derivatives.

The present thesis is structured into three chapters .
chapter 1 , contains the definition of functional spaces then we moved on to the left and right fractional operators and their properties . [24]
In the chapter 2 , we study the Cauchy problem of fractional order with a polynomial nonlinearity:

$$(D_{0+}^{\alpha,\beta}u)(t) \geq t^{\delta}|u(t)|^{m}, \quad t>0, \quad m>1, \quad \delta \in \mathbb{R} \tag{1}$$

and we present some definitions, lemmas, properties, and notation that will be used in our results. [6]

Chapter 3, we study the non existence of nontrivial global solutions for a system of fractional differential equations with a nonlinear source term and two fractional derivatives of different orders. More precisely, we consider the following system:

$$\begin{cases} D_0^{\alpha_1}u_1(\tau) = f_1\left[\tau, D_0^{\rho_1}u_1(\tau), D_0^{\rho_2}u_2(\tau)\right], \\ D_0^{\alpha_2}u_2(\tau) = f_2\left[\tau, D_0^{\rho_1}u_1(\tau), D_0^{\rho_2}u_2(\tau)\right], \end{cases} \qquad \tau > 0 \tag{2}$$

Finally, this study ends with a general conclusion.

CHAPTER 1

BASIC CONCEPTS

1.1 Functional Spaces

Let $0 \leq \gamma < 1$. We introduce the following functional spaces:

$$C^{\gamma}[a,b] = \{f(x) : (a,b] \to \mathbb{R} \,|\, (x-a)^{\gamma} f(x) \in C[a,b]\},$$

which is a Banach space with the norm

$$\|f\|_{C^{\gamma}} = \sup_{x \in [a,b]} |(x-a)^{\gamma} f(x)|.$$

$$C^{n,\gamma}[a,b] = \left\{ f \in C^{n-1}[a,b] \,\middle|\, f^{(n)} \in C^{\gamma}[a,b] \right\},$$

which is also a Banach space with the norm

$$\|f\|_{C^{n,\gamma}} = \sum_{k=0}^{n-1} \|f^{(k)}\|_{\infty} + \|f^{(n)}\|_{C^{\gamma}}, \quad n \in \mathbb{N}.$$

Moreover, $C^{0,\gamma}[a,b] = C^{\gamma}[a,b]$.

1.2 Left and Right Fractional Operators and Their Properties

In this subsection, we introduce the definitions of the left and right fractional operators of Riemann–Liouville, along with their main properties.

Riemann–Liouville Fractional Integral

Let f be a continuous function on the interval $[a,b]$. We consider the following integrals:

$$I^{(1)}f(t)=\int_a^t f(\tau)\,d\tau, \tag{1.1}$$

$$I^{(2)}f(t)=\int_a^t\left(\int_a^{t_1} f(\tau)\,d\tau\right)dt_1. \tag{1.2}$$

According to Fubini's theorem, we obtain:

$$I^{(2)}f(t)=\frac{1}{1!}\int_a^t (t-\tau)^{2-1}f(\tau)\,d\tau. \tag{1.3}$$

By repeating this process n times, we get:

$$I^{(n)}f(t)=\int_a^t dt_1\int_a^{t_1}dt_2\cdots\int_a^{t_{n-1}} f(\tau)\,d\tau \tag{1.4}$$

$$=\frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1}f(\tau)\,d\tau, \tag{1.5}$$

for any integer n. This formula is called the **Cauchy formula**. Since $(n-1)!=\Gamma(n)$, Riemann realized that the last expression could also make sense when n takes non-integer values.

It was therefore natural to define the fractional integration operator as follows:

Definition 1. *Let $f\in L^1([a,+\infty[)$, with $a\in\mathbb{R}$ and $\alpha\in\mathbb{R}_+^*$. The left and right Riemann–Liouville fractional integrals are defined as follows:*

The Liouville fractional integral of order α is defined by:

$$I_{a+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)\,d\tau,\quad a<t,$$

and

$$I_{b-}^{\alpha}f(t)=\frac{-1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}f(\tau)\,d\tau,\quad t<b.$$

Moreover, we have:

$$I_{a+}^{0}f(t)=I_{b-}^{0}f(t)=f(t),$$

i.e., I_{a+}^{0} is the identity operator.

Remark 2. *By the change of variable* $s = t - \tau$*, we observe that* I^{α}_{a+} *can be written in the following form:*

$$I^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^{t-a} s^{\alpha-1} f(t-s)\, ds.$$

Definition 3. . *Leibniz's Rule for Differentiation is defined by:*

$$\frac{d}{dt}\left(\int_0^{b(t)} f(t,x)\, dx\right) = f(t,b(t)) \cdot b'(t) + \int_0^{b(t)} \frac{\partial}{\partial t} f(t,x)\, dx$$

Riemann–Liouville Fractional Integrals of Some Common Functions

Property 1. *If* $\alpha \geq 0$ *and* $\beta > 0$*, then:*

$$I^{\alpha}_{a+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1},$$

$$I^{\alpha}_{b-}(b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-t)^{\beta+\alpha-1}.$$

Proof. 1. Let $f(t) = (t-a)^{\beta-1}$, $t > a$, with $a \in \mathbb{R}$ and $\beta > 0$:

$$I^{\alpha}_{a_+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - a)^{\beta}\, d\tau.$$

Using the change of variable $\tau = a + (t-a)s$, where s varies from 0 to 1, and the Beta function, we get:

$$\begin{aligned} I^{\alpha}_{a_+} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^1 [t-a-(t-a)s]^{\alpha-1} [s(t-a)]^{\beta} (t-a)\, ds \\ &= \frac{1}{\Gamma(\alpha)} (t-a)^{\alpha+\beta} \int_0^1 s^{\beta}(1-s)^{\alpha-1}\, ds \\ &= \frac{1}{\Gamma(\alpha)} (t-a)^{\alpha+\beta} B(\beta+1,\alpha) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (t-a)^{\alpha+\beta}. \end{aligned}$$

Hence,

$$I^{\alpha}_{a_+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta}. \tag{1.6}$$

For $a = 0$, we have:

$$I_{0_+}^{\alpha} t^{\beta} = I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \tag{1.7}$$

By the same method we can show that

$$I_{b_-}^{\alpha} (b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (b-t)^{\beta+\alpha-1}.$$

Note that the fractional integral of the constant function $f(t) = C$ is demonstrated as follows

$$\begin{aligned} I_{a_+}^{\alpha} C &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} C \, d\tau \\ &= \frac{C}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \, d\tau \\ &= \frac{C}{\Gamma(\alpha)} \left(\frac{-(t-\tau)^{\alpha}}{\alpha} \right) \Big|_a^t \\ &= \frac{C}{\alpha\Gamma(\alpha)} (t-a)^{\alpha} \\ &= \frac{C}{\Gamma(\alpha+1)} (t-a)^{\alpha}. \end{aligned}$$

Thus,

$$I_{a_+}^{\alpha} C = \frac{C}{\Gamma(\alpha+1)} (t-a)^{\alpha}. \tag{1.8}$$

□

Property 2. *If* $\alpha > 0$ *and* $\beta > 0$*, then the following equations hold:*

$$I_{a+}^{\alpha} \left(I_{a+}^{\beta} f(t) \right) = I_{a+}^{\alpha+\beta} f(t), \quad I_{b-}^{\alpha} \left(I_{b-}^{\beta} f(t) \right) = I_{b-}^{\alpha+\beta} f(t).$$

These relations are satisfied at almost every point $t \in [a,b]$ *for* $f(t) \in L^p([a,b], \mathbb{R}^N)$*, with* $1 \leq p < \infty$*.*

If $\alpha + \beta > 1$*, then the above relations hold at every point of* $[a,b]$*.*

Properties of the Riemann-Liouville Fractional Integral

In this section we study the properties of the left fractional integral of Riemann-liouville then we will do the same thing on the right side

Theorem 4. *If* $f \in L^1[a,b]$ *and* $\alpha > 0$, *then* $I^{\alpha}_{a+} f(t)$ *exists for almost every* $t \in [a,b]$, *and we have:*

$$I^{\alpha}_{a+} f \in L^1[a,b].$$

Proof. Soit $f \in L^1[a,b]$, on a :

$$I^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau = \int_{-\infty}^{+\infty} g(t-\tau) h(\tau)\, d\tau,$$

avec $-\infty \leq a < t < +\infty$, tel que :

$$g(u) = \begin{cases} \frac{u^{\alpha-1}}{\Gamma(\alpha)}, & \text{si } 0 < u \leq b-a, \\ 0, & \text{si } u \in \mathbb{R} \setminus (0, b-a), \end{cases}$$

et

$$h(u) = \begin{cases} f(u), & \text{si } a \leq u \leq b, \\ 0, & \text{si } u \in \mathbb{R} \setminus [a,b]. \end{cases}$$

Comme $g, h \in L^1(\mathbb{R})$, alors $I^{\alpha}_{a+} f \in L^1[a,b]$. □

Theorem 5. *For* $f \in L^1[a,b]$, *the Riemann-Liouville fractional integral satisfies the following semigroup property:*

$$I^{\alpha}_{a_+}\left(I^{\beta}_{a_+} f\right)(t) = I^{\alpha+\beta}_{a_+} f(t),$$

for $\alpha > 0$ *and* $\beta > 0$.

Proof. Let $f \in L^1[a,b]$, with $\alpha > 0$ and $\beta > 0$. Then we have:

$$\begin{aligned} I^{\alpha}_{a_+}\left(I^{\beta}_{a_+} f\right)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \left(I^{\beta}_{a_+} f\right)(\tau)\, d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \left[\frac{1}{\Gamma(\beta)} \int_a^{\tau} (\tau-s)^{\beta-1} f(s)\, ds\right] d\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-\tau)^{\alpha-1} \left[\int_a^{\tau} (\tau-s)^{\beta-1} f(s)\, ds\right] d\tau. \end{aligned}$$

□

Remark 6. *The Riemann-Liouville fractional integral can also be expressed as a convolution product of the power function $h_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ with the function $f(t)$:*

$$I^{\alpha}_{a_+} f(t) = \int_a^t h_\alpha(t-\tau) f(\tau)\, d\tau = (h_\alpha * f)(t).$$

Proposition 7. *The operator $I^{\alpha}_{a_+}$ is linear.*

Proof. Indeed, if f and g are two functions such that $I^{\alpha}_{a_+} f$ and $I^{\alpha}_{a_+} g$ exist, then for any real numbers c_1 and c_2, we have:

$$\begin{aligned} I^{\alpha}_{a_+}(c_1 f + c_2 g)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (c_1 f + c_2 g)(\tau)\, d\tau \\ &= \frac{c_1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau + \frac{c_2}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} g(\tau)\, d\tau \\ &= c_1 I^{\alpha}_{a_+} f(t) + c_2 I^{\alpha}_{a_+} g(t). \end{aligned}$$

□

Proposition 8. *Let $f \in C([a,b])$. Then we have:*

1. $\frac{d}{dt}\left(I^{\alpha}_{a_+} f\right)(t) = \left(I^{\alpha-1}_{a_+} f\right)(t), \quad \alpha > 1.$

2. $\lim_{\alpha\to 0^+}\left(I^{\alpha}_{a_+} f\right)(t) = f(t), \quad \alpha > 0.$

Proof. 1. Let us apply the Leibniz rule for differentiation (definition 2). We obtain:

$$\begin{aligned} \frac{d}{dt}\left(I^{\alpha}_{a_+} f\right)(t) &= \frac{d}{dt}\left(\frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau\right) \\ &= \frac{\alpha-1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-2} f(\tau)\, d\tau \\ &= \frac{\alpha-1}{\Gamma((\alpha-1)+1)} \int_a^t (t-\tau)^{\alpha-2} f(\tau)\, d\tau \\ &= \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-\tau)^{\alpha-2} f(\tau)\, d\tau \\ &= \left(I^{\alpha-1}_{a_+} f\right)(t). \end{aligned}$$

2. For the second identity, since $f \in C([a,b])$, we have:

$$I^{\alpha}_{a_+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau.$$

According to relation (remark 2), we can write:

$$I_{a_+}^{\alpha} 1 = \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \longrightarrow 1$$

as $\alpha \to 0^+$. Thus, for some $\delta > 0$, we have:

$$\begin{aligned}
\left| I_{a_+}^{\alpha} f(t) - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} f(t) \right| &= \left| \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau - \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(t)\, d\tau \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} |f(\tau) - f(t)|\, d\tau \qquad (1.9) \\
&= \frac{1}{\Gamma(\alpha)} \int_a^{t-\delta} (t-\tau)^{\alpha-1} |f(\tau) - f(t)|\, d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t-\delta}^{t} (t-\tau)^{\alpha-1} |f(\tau) - f(t)|\, d\tau. \qquad (1.10)
\end{aligned}$$

On one hand, since f is continuous on $[a,b]$, we have:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall t, \tau \in [a,b] \text{ with } |\tau - t| < \delta \Rightarrow |f(\tau) - f(t)| < \varepsilon.$$

This implies:

$$\int_{t-\delta}^{t} (t-\tau)^{\alpha-1} |f(\tau) - f(t)|\, d\tau \leq \varepsilon \int_{t-\delta}^{t} (t-\tau)^{\alpha-1}\, d\tau = \frac{\varepsilon \delta^{\alpha}}{\alpha}. \qquad (1.11)$$

On the other hand:

$$\begin{aligned}
\int_a^{t-\delta} (t-\tau)^{\alpha-1} |f(\tau) - f(t)|\, d\tau &\leq \frac{1}{\Gamma(\alpha)} \int_a^{t-\delta} (t-\tau)^{\alpha-1} \left(|f(\tau)| + |f(t)|\right) d\tau \qquad (1.12) \\
&\leq 2 \sup_{\xi \in [a,t]} |f(\xi)| \int_a^{t-\delta} (t-\tau)^{\alpha-1}\, d\tau \qquad (1.13) \\
&= 2M \left(\frac{(t-a)^{\alpha}}{\alpha} - \frac{\delta^{\alpha}}{\alpha} \right), \qquad (1.14)
\end{aligned}$$

where $M = \sup_{\xi \in [a,t]} |f(\xi)|$.

Combining (1.9), (1.11), and (1.12), we get:

$$\begin{aligned}
\left| I_{a+}^{\alpha} f(t) - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} f(t) \right| &\leq \frac{1}{\alpha \Gamma(\alpha)} \left[\varepsilon \delta^{\alpha} + 2M \left((t-a)^{\alpha} - \delta^{\alpha} \right) \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[\varepsilon \delta^{\alpha} + 2M \left((t-a)^{\alpha} - \delta^{\alpha} \right) \right].
\end{aligned}$$

Letting $\alpha \to 0^+$, we obtain:

$$\left|I_{a_+}^{\alpha} f(t) - f(t)\right| \leq \frac{\varepsilon}{\Gamma(\alpha+1)}, \quad \forall \varepsilon > 0,$$

which shows that:

$$\lim_{\alpha \to 0^+} I_{a_+}^{\alpha} f(t) = f(t).$$

Letting $\alpha \to 0^+$, we obtain:

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which shows that:

$$\lim_{\alpha \to 0^+} I_{a_+}^{\alpha} f(t) = f(t).$$

□

The following lemmas provide some properties of $I_{a^+}^{\alpha}$.

Lemma 9. *For $\alpha > 0$, $I_{a^+}^{\alpha}$ maps $C[a,b]$ into $C[a,b]$.*

Lemma 10. *Let $\alpha > 0$ and $0 \leq \gamma < 1$. Then $I_{a^+}^{\alpha}$ is bounded from $C_\gamma[a,b]$ into $C_\gamma[a,b]$.*

Lemma 11. *Let $\alpha > 0$ and $0 \leq \gamma < 1$. If $\gamma \leq \alpha$, then $I_{a^+}^{\alpha}$ is bounded from $C_\gamma[a,b]$ into $C[a,b]$.*

Lemma 12. *Let $0 \leq \gamma < 1$ and $f \in C_\gamma[a,b]$. Then*

$$I_{a^+}^{\alpha} f(a) = \lim_{x \to a^+} I_{a^+}^{\alpha} f(x) = 0, \textit{ for } 0 \leq \gamma < \alpha.$$

Proof. (lemma 12)
Note that, according to Lemma 11, $I_{a^+}^{\alpha} f \in C_\gamma[a,b]$. Since $f \in C_\gamma[a,b]$, it follows that $(x-a)^\gamma f(x)$ is continuous on $[a,b]$. Therefore, there exists a constant $M > 0$ such that

$$|(x-a)^\gamma f(x)| < M, \quad \text{for all } x \in [a,b].$$

Hence,

$$\left|I_{a^+}^{\alpha} f(x)\right| < M\left[I_{a^+}^{\alpha}(t-a)^{-\gamma}\right].$$

By the lemma, we have

$$\left|I_{a^+}^{\alpha} f(x)\right| < M \cdot \frac{\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)}(x-a)^{\alpha-\gamma}.$$

Since $\alpha > \gamma$, the right-hand side tends to 0 as $x \to a^+$. □

1.2.1 Left and Right Fractional derivatives and Their Properties

Definition 13. *(Left and right Riemann-Liouville fractional derivatives.)*
The left and right Riemann-Liouville fractional derivatives $D_{a_+}^{\alpha} f$ and $D_{b_-}^{\alpha} f$ of order $\alpha \in \mathbb{R}^+$ are defined by

$$D_{a_+}^{\alpha} f(t) = \frac{d^n}{dt^n}\left(I_{a_+}^{(n+\alpha)} f(t)\right) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1} f(s)\,ds, \quad t > a,$$

and

$$D_{b_-}^{\alpha} f(t) = (-1)^n \frac{d^n}{dt^n}\left(I_{b_-}^{(n+\alpha)} f(t)\right) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (s-t)^{n-\alpha-1} f(s)\,ds, \quad t < b,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

In particular, when $\alpha = n \in \mathbb{N}_0$, we have:

$$I_{a_+}^{0} f(t) = I_{b_-}^{0} f(t) = f(t),$$

$$I_{a_+}^{n} f(t) = f^{(n)}(t), \quad \textit{and} \quad I_{b_-}^{n} f(t) = (-1)^n f^{(n)}(t),$$

where $f^{(n)}(t)$ denotes the usual derivative of order n.

If $0 < \alpha < 1$, then:

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-\alpha} f(s)\,ds, \quad t > a,$$

$$I_{b-}^{\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b (s-t)^{-\alpha} f(s)\,ds, \quad t < b.$$

Remark 14. *If $f \in C([a,b],\mathbb{R}^N)$, it is clear that the Riemann-Liouville fractional integral of order $\alpha > 0$ exists on $[a,b]$. On the other hand, following Kilbas et al.[13], the Riemann-Liouville fractional derivative of order $\alpha \in [n-1,n)$ exists almost everywhere on $[a,b]$ if $f \in AC^n([a,b],\mathbb{R}^N)$.*

The left and right Caputo fractional derivatives are defined via the dabove Riemann-Liouville fractional derivatives

Definition 15. *(Left and right Caputo fractional derivatives).*
The left and right Caputo fractional derivatives ${}^{C}D^{\alpha}_{a-}f(t)$ *and* ${}^{C}D^{\alpha}_{b\_}f(t)$ *of order* $\alpha \in \mathbb{R}^{+}$ *are defined by*

$$
{}^{C}D^{\alpha}_{a+}f(t) = D^{\alpha}_{a+}\left(f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^{k}\right),
$$

$$
{}^{C}D^{\alpha}_{b\_}f(t) = D^{\alpha}_{b\_}\left(f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(b-t)^{k}\right),
$$

respectively, where

$$
n = \begin{cases} [\alpha]+1 & \text{if } \alpha \notin \mathbb{N}_0, \\ \alpha & \text{if } \alpha \in \mathbb{N}_0. \end{cases} \tag{1.3}
$$

In particular, when $0 < \alpha < 1$*, then:*

$$
{}^{C}D^{\alpha}_{a+}f(t) = D^{\alpha}_{a+}(f(t) - f(a)),
$$

$$
{}^{C}D^{\alpha}_{b-}f(t) = D^{\alpha}_{b-}(f(t) - f(b)).
$$

The Riemann-Liouville and Caputo fractional derivatives are connected by the following relations.

Property 3. *(i) If* $\alpha \in \mathbb{N}_0$ *and* $f(t)$ *is a function for which the Caputo fractional derivatives* ${}^{C}D^{\alpha}_{a+}f(t)$ *and* ${}^{C}D^{\alpha}_{b-}f(t)$ *of order* $\alpha \in \mathbb{R}^{+}$ *exist, together with the Riemann-Liouville fractional derivatives* $D^{\alpha}_{a+}f(t)$ *and* $D^{\alpha}_{b-}f(t)$*, then:*

$$
{}^{C}D^{\alpha}_{a+}f(t) = D^{\alpha}_{a+}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha},
$$

$$
{}^{C}D^{\alpha}_{b-}f(t) = D^{\alpha}_{b-}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha},
$$

where $n = [\alpha]+1$.

In particular, when $0 < \alpha < 1$*, we have:*

$$
{}^{C}D^{\alpha}_{a+}f(t) = D^{\alpha}_{a+}f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha},
$$

$$
{}^{C}D^{\alpha}_{b-}f(t) = D^{\alpha}_{b-}f(t) - \frac{f(b)}{\Gamma(1-\alpha)}(b-t)^{-\alpha}.
$$

(ii) If $\alpha = n \in \mathbb{N}_0$ and the classical derivative $f^{(n)}(t)$ of order n exists, then the Caputo derivatives reduce to:

$$^{C}D^{n}_{a+}f(t) = f^{(n)}(t), \quad \text{and} \quad ^{C}D^{n}_{b-}f(t) = (-1)^{n} f^{(n)}(t).$$

Property 4. *Let $\alpha \in \mathbb{R}^{+}$ and let n be given by (1.3). If $f(t) \in AC^{n}([a,b],\mathbb{R}^{N})$, then the Caputo fractional derivatives $^{C}D^{\alpha}_{a+}f(t)$ and $^{C}D^{\alpha}_{b-}f(t)$ exist almost everywhere on $[a,b]$.*

(i) If $\alpha \in \mathbb{N}_0$, the Caputo derivatives are represented by

$$^{C}D^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds\right)$$

and

$$^{C}D^{\alpha}_{b-}f(t) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\left(\int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s)\,ds\right),$$

where $n = [\alpha]+1$.

In particular, when $0<\alpha<1$ and $f(t) \in AC([a,b],\mathbb{R}^{N})$, we have:

$$^{C}D^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha} f'(s)\,ds\right),$$

and

$$^{C}D^{\alpha}_{b-}f(t) = -\frac{1}{\Gamma(1-\alpha)}\left(\int_t^b (s-t)^{-\alpha} f'(s)\,ds\right).$$

(ii) If $\alpha = n \in \mathbb{N}_0$, then $^{C}D^{\alpha}_{a+}f(t)$ and $^{C}D^{\alpha}_{b-}f(t)$ are given by (1.4). In particular,

$$^{C}D^{0}_{a+}f(t) = {}^{C}D^{0}_{b-}f(t) = f(t).$$

Property 5. *If $\alpha \geq 0$ and $\beta > 0$, then*

$$D^{-\alpha}_{a+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}, \quad \alpha > 0,$$

$$D^{\alpha}_{a+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \quad \alpha \geq 0,$$

$$D^{-\alpha}_{b-}(b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-t)^{\beta+\alpha-1}, \quad \alpha > 0,$$

$$D^{\alpha}_{b-}(b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1}, \quad \alpha \geq 0.$$

In particular, if $\beta = 1$ and $\alpha \geq 0$, then the Riemann–Liouville fractional derivatives of a constant are, in general, not equal to zero:

$$D^{\alpha}_{a+}1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad D^{\alpha}_{b-}1 = \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0<\alpha<1.$$

On the other hand, for $j = 1, 2, \ldots, [\alpha]+1$,

$$D^{\alpha}_{a+}(t-a)^{\alpha-j} = 0, \quad D^{\alpha}_{b-}(b-t)^{\alpha-j} = 0.$$

The semigroup properties of the fractional integration operators $D^{-\alpha}_{a+}$ and $D^{-\alpha}_{b-}$ are given by the following results.

Property 6. *(i) If $\alpha > 0$ and $f(t) \in L^p([a,b],\mathbb{R}^N)$ with $1 \leq p \leq \infty$, then the following equalities hold almost everywhere on $[a,b]$:*

$$D^{\beta}_{a_+}\left(D^{-\alpha}_{a_+}f(t)\right) = f(t), \quad \text{and} \quad D^{\alpha}_{b_-}\left(D^{-\alpha}_{b}f(t)\right) = f(t).$$

(ii) If $\alpha > \beta > 0$, and $f(t) \in L^p([a,b],\mathbb{R}^N)$ with $1 \leq p \leq \infty$, then the following relations hold almost everywhere on $[a,b]$:

$$D^{\beta}_{a_+}\left(D^{-\alpha+\beta}_{a_+}f(t)\right) = f(t), \quad \text{and} \quad D^{\beta}_{b_-}\left(D^{\alpha}_{b_-}f(t)\right) = D^{-\alpha+\beta}_{b_-}f(t).$$

In particular, when $\beta = k \in \mathbb{N}^+$ and $\alpha > k$, we have:

$$D^{k}_{a_+}\left(D^{-\alpha+k}_{a_+}f(t)\right) = D^{-\alpha+k}_{a_+}f(t), \quad \text{and} \quad D^{-\alpha}_{b_-}\left(D^{-\alpha+k}_{b_-}f(t)\right) = (-1)^k D^{-\alpha+k}_{b_-}f(t).$$

To present the next property, we use the function spaces $I^{\alpha}_{a_+}(L^p)$ and $I^{\alpha}_{b_-}(L^p)$, defined for $\alpha > 0$ and $1 \leq p \leq \infty$ by

$$I^{\alpha}_{a_+}(L^p) = \left\{f : f = D_{a_+}It^{\alpha}\varphi,\ \varphi \in L^p([a,b],\mathbb{R}^N)\right\}$$

and

$$I^{\alpha}_{b_-}(L^p) = \left\{f : f = {}_tD^{-\alpha}_{b}\varphi,\ \varphi \in L^p([a,b],\mathbb{R}^N)\right\},$$

1.2.2 Hilfer Fractional derivative

Definition 16. *The Hilfer fractional derivative of order $0 < \alpha < 1$, and type $0 \leq \beta \leq 1$, of a function $f(\cdot)$ is defined as:*

$$D^{\alpha,\beta}_{a+}f(x) = \left(I^{\beta(1-\alpha)}_{a+}D\left(I^{(1-\beta)(1-\alpha)}_{a+}f\right)\right)(x),$$

where

$$D = \frac{d}{dx}.$$

The Hilfer fractional derivative is considered an interpolator between the Riemann–Liouville and Caputo derivatives. The following remarks illustrate the relation with Caputo and Riemann–Liouville operators.

Remark 17. *[5] We have*

(i) The operator $D_{a+}^{\alpha,\beta}$ *can also be written as:*

$$D_{a+}^{\alpha,\beta} = I_{a+}^{\beta(1-\alpha)} D I_{a+}^{(1-\beta)(1-\alpha)} = I_{a+}^{\beta(1-\alpha)} D_{a+}^{\gamma}, \quad \textit{with } \gamma = \alpha + \beta - \alpha\beta.$$

(ii) If $\beta = 0$, *we obtain the Riemann–Liouville fractional derivative:*

$$D_{a+}^{\alpha} = D_{a+}^{\alpha,0}.$$

(iii) If $\beta = 1$, *we obtain the Caputo fractional derivative:*

$${}^{C}D_{a+}^{\alpha} = I_{a+}^{1-\alpha} D.$$

CHAPTER 2
NON-EXISTENCE OF GLOBAL SOLUTIONS FOR A DIFFERENTIAL EQUATION INVOLVING HILFER FRACTIONAL DERIVATIVE

Introduction

In this chapter [6], we study the Cauchy problem of fractional order with a polynomial nonlinearity:

$$(D_{0+}^{\alpha,\beta}u)(t) \geq t^{\delta}|u(t)|^{m}, \quad t>0, \quad m>1, \quad \delta\in\mathbb{R} \tag{2.1}$$

with initial condition:

$$(D_{0+}^{\gamma-1}u)(0) = b > 0, \tag{2.2}$$

where

$$(D_{0+}^{\alpha,\beta}y)(x) = \left(I_{0+}^{\beta(1-\alpha)}\frac{d}{dx}\left(I_{0+}^{(1-\beta)(1-\alpha)}f\right)\right)(x) \tag{2.3}$$

is the Hilfer fractional derivative (HFD) of order $0<\alpha<1$ and type $0\leq\beta\leq 1$, with

$$\gamma = \alpha+\beta-\alpha\beta,$$

and I_{0+}^{σ}, for $\sigma>0$, is the standard Riemann–Liouville fractional integral of order σ. This type of derivatives were introduced by Hilfer in [5] [21] These references provide information on the applications and origins of this operator. It is easy to observe that this derivative interpolates between the Riemann–Liouville fractional derivative (for $\beta=0$) and the Caputo fractional derivative (for $\beta=1$) [13] [20]. The special case $\beta=0$ has been discussed in [23]. In this chapter, we determine the range of values of m for which solutions to the problem do not exist globally. We also establish an optimal exponent (in a certain sense), by showing that solutions exist beyond this bound in an appropriate function space.

2.1 Preliminaries

In this section, we present some definitions, lemmas, properties, and notation that will be used in our results.

Definition 18 ([13])**.** *[Weighted Space] Let $\Omega = [a,b]$ be a finite interval and $0 \leq \gamma < 1$. We introduce the weighted space $C_\gamma[a,b]$ of continuous functions f on $(a,b]$ defined as*

$$C_\gamma[a,b] = \{f : (a,b] \to \mathbb{R} \mid (x-a)^\gamma f(x) \in C[a,b]\}.$$

The norm in the space $C_\gamma[a,b]$ is given by

$$\|f\|_{C_\gamma} = \|(x-a)^\gamma f(x)\|_C, \quad C_0[a,b] = C[a,b].$$

Definition 19 ([13])**.** *[Riemann–Liouville Left-Sided Fractional Integral] The Riemann–Liouville left-sided fractional integral $I^\alpha_{a+} f$ of order $\alpha > 0$ is defined by*

$$(I^\alpha_{a+} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad (a < x \leq b,\ \alpha > 0),$$

provided that the integral exists. Here, $\Gamma(\alpha)$ is the Gamma function. When $\alpha = 0$, we define $I^0_{a+} f = f$. In fact, it can be shown that $I^\alpha_{a+} f \to f$ as $\alpha \to 0$.

Definition 20 ([13])**.** *[Riemann–Liouville Left-Sided Fractional Derivative] The Riemann–Liouville left-sided fractional derivative $D^\alpha_{a+} f$ of order $\alpha \in [0,1)$ is defined by*

$$(D^\alpha_{a+} f)(x) = \frac{d}{dx}\left(I^{1-\alpha}_{a+} f\right)(x),$$

that is,

$$(D^\alpha_{a+} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha}\, dt, \quad (x > a,\ 0 < \alpha < 1).$$

When $\alpha = 1$, we have $D^1_{a+} f = Df$, and when $\alpha = 0$, we define $D^0_{a+} f = f$.

Definition 21 ([13])**.** *[Riemann–Liouville Right-Sided Fractional Derivative] The Riemann–Liouville right-sided fractional derivative $D^\alpha_{b-} f$ of order $\alpha \in [0,1)$ is defined by*

$$(D^\alpha_{b-} f)(x) = -\frac{d}{dx}\left(I^{1-\alpha}_{b-} f\right)(x),$$

that is,

$$(D^\alpha_{b-} f)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(t)}{(t-x)^\alpha}\, dt, \quad (a \leq x < b,\ 0 < \alpha < 1).$$

In particular, when $\alpha = 0$, we have $D^0_{b-} f = f$ [Space $C^{1-\gamma}_\gamma[a,b]$] We define the space

$$C^{1-\gamma}_\gamma[a,b] = \left\{y \in C^{1-\gamma}[a,b] \,\middle|\, D^\gamma_{a+} y \in C^{1-\gamma}[a,b]\right\}.$$

Definition 22 ([13]). *(Space $C_\gamma^{1-\gamma}[a,b]$) We define the space*

$$C_\gamma^{1-\gamma}[a,b] = \left\{ y \in C^{1-\gamma}[a,b] \mid D_{a+}^{\gamma} y \in C^{1-\gamma}[a,b] \right\}.$$

Lemma 23 ([22]). *Let $0 < \alpha < 1$ and $0 \le \gamma < 1$. If $f \in C_\gamma^1$, the space of continuous functions on $[a,b]$ whose derivatives belong to C_γ, then the fractional derivatives $D_{a+}^{\alpha} f$ and $D_{b-}^{\alpha} f$ exist on $(a,b]$ and $[a,b)$, respectively, and can be represented as:*

$$(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(a)}{(x-a)^{\alpha}} + \int_a^x \frac{f'(t)}{(x-t)^{\alpha}} \, dt \right],$$

$$(D_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(b)}{(b-x)^{\alpha}} - \int_x^b \frac{f'(t)}{(t-x)^{\alpha}} \, dt \right].$$

Lemma 24 ([19]). *[Semigroup Property of Fractional Integration.] Let $\alpha > 0$, $\beta > 0$, and $0 \le \gamma < 1$. If $f \in L^p(a,b)$, for $1 \le p \le \infty$, then the equation*

$$I_{a+}^{\alpha} I_{a+}^{\beta} f = I_{a+}^{\alpha+\beta} f$$

holds almost everywhere on $[a,b]$. When $\alpha + \beta > 1$, this relation holds at every point $x \in [a,b]$.

Lemma 25 ([19]). *[Fractional Integration by Parts.] Let $\alpha > 0$, $p \ge 1$, $q \ge 1$, and $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ (with $p \ne 1$, $q \ne 1$ when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $\varphi \in L^p(a,b)$ and $\psi \in L^q(a,b)$, then*

$$\int_a^b \varphi(x)(I_{a+}^{\alpha} \psi)(x) \, dx = \int_a^b \psi(x)(I_{b-}^{\alpha} \varphi)(x) \, dx.$$

Definition 26 (Caputo Fractional Derivative). *The Caputo fractional derivative ${}^C D_{a+}^{\alpha} f$ of order $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, on $[a,b]$ is defined by*

$${}^C D_{a+}^{\alpha} f = I_{a+}^{1-\alpha} D f,$$

where $D = \frac{d}{dx}$.

Theorem 27 (Young's Inequality). *If a and b are nonnegative real numbers and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

2.2 Non-existence Result

In this chapter, we establish sufficient conditions ensuring the nonexistence of global solutions. In particular, we determine a range of values for the exponent m for which solutions cannot be continued for all time. The proof is mainly based on the test function method developed by Mitidieri and Pohozaev [17], along with suitable manipulations of fractional derivatives and integrals. In addition to the results stated in the Preliminaries section, we require the following lemma.

Lemma 28. *If $\alpha > 0$ and $f \in C[a,b]$, then*

$$(I_{a+}^{\alpha} f)(a) = \lim_{t \to a} (I_{a+}^{\alpha} f)(t) = 0, \quad \text{and} \quad (I_{b-}^{\alpha} f)(b) = \lim_{t \to b} (I_{b-}^{\alpha} f)(t) = 0.$$

Proof. Since $f \in C[a,b]$, we have $|f(t)| < M$ on $[a,b]$ for some positive constant M. Therefore,

$$|(I_{a+}^{\alpha} f)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |f(s)| ds \leq \frac{M}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} ds = \frac{M}{\alpha\Gamma(\alpha)} (t-a)^{\alpha}.$$

Since $\alpha > 0$, it follows that:

$$\lim_{t \to a} (I_{a+}^{\alpha} f)(t) = 0.$$

The second identity is proved similarly. □ □

Theorem 29. *Assume that $\delta > -\alpha$ and $1 < m \leq \dfrac{\delta+1}{1-\alpha}$. Then, Problem (1.1) does not admit global nontrivial solutions in $C_{1-\gamma}^{\gamma}$ when $b > 0$.*

Proof. Assume, on the contrary, that a nontrivial solution u exists for all $t > 0$. Let $\varphi \in C^1([0,\infty))$ be a nonnegative, non-increasing test function such that:

$$\varphi(t) := \begin{cases} 1, & t \in [0, T/2], \\ 0, & t \in [T, \infty), \end{cases}$$

for some $T > 0$. Multiplying the inequality in (1.1) by $\varphi(t)$ and integrating over $[0,T]$ yields:

$$\int_0^T (D_{0+}^{\alpha,\beta} u)(t) \varphi(t)\, dt \geq \int_0^T t^{\delta} |u(t)|^m \varphi(t)\, dt. \tag{3.1}$$

From the definition of $(D_{0+}^{\alpha,\beta} u)(t)$ (see (1.2)), we can write:

$$\int_0^T I_{0+}^{\beta(1-\alpha)} \frac{d}{dt} \left(I_{0+}^{1-\gamma} u \right)(t)\, \varphi(t)\, dt \geq \int_0^T t^{\delta} |u(t)|^m\, \varphi(t)\, dt. \tag{3.2}$$

By Lemma 7, we can deduce from (3.2) that:

$$\int_0^T \frac{d}{dt}\left(I_{0+}^{1-\gamma}u\right)(t)\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(t)\,dt \geq \int_0^T t^{\delta}|u(t)|^m \varphi(t)\,dt.$$

An integration by parts yields:

$$\left[\left(I_{0+}^{1-\gamma}u\right)(t)\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(t)\right]_{t=0}^{T} - \int_0^T \left(I_{0+}^{1-\gamma}u\right)(t)\frac{d}{dt}\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(t)\,dt \geq \int_0^T t^{\delta}|u(t)|^m\varphi(t)\,dt.$$

Using Lemma 8, we know that $\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(T) = 0$ and

$$\left(I_{0+}^{1-\gamma}u\right)(0) = \left(D_{0+}^{\gamma-1}u\right)(0) = b,$$

so we get:

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(0) - \int_0^T \left(I_{0+}^{1-\gamma}u\right)(t)\frac{d}{dt}\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(t)\,dt \geq \int_0^T t^{\delta}|u(t)|^m\varphi(t)\,dt.$$

From Definition 9, it follows that:

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(0) + \int_0^T \left(I_{0+}^{1-\gamma}u\right)(t)\left(D_{T-}^{1-\beta(1-\alpha)}\varphi\right)(t)\,dt \geq \int_0^T t^{\delta}|u(t)|^m\varphi(t)\,dt.$$

And from Lemma 5, we have:

$$\begin{aligned} -b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(0) + \int_0^T \left(I_{0+}^{1-\gamma}u\right)(t)&\left[\frac{1}{\Gamma[\beta(1-\alpha)]}\left(\frac{\varphi(T)}{(T-t)^{1-\beta(1-\alpha)}}\right.\right. \\ &\left.\left. - \int_t^T \frac{\varphi'(s)}{(s-t)^{1-\beta(1-\alpha)}}ds\right)\right]dt \\ &\geq \int_0^T t^{\delta}|u(t)|^m\varphi(t)\,dt. \end{aligned} \tag{3.4}$$

Since $\varphi(T) = 0$, relation (6) becomes:

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(0) - \int_0^T \left(I_{0+}^{1-\gamma}u\right)(t)\left(I_{T-}^{\beta(1-\alpha)}\varphi'\right)(t)\,dt \geq \int_0^T t^{\delta}|u(t)|^m\varphi(t)\,dt.$$

Lemma 7 allows us to write:

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(0) - \int_0^T \varphi'(t)\left(I_{0+}^{\beta(1-\alpha)}I_{0+}^{1-\gamma}u\right)(t)\,dt \geq \int_0^T t^{\delta}|u(t)|^m\varphi(t)\,dt,$$

and by Lemma 6:

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(0)-\int_0^T\varphi'(t)\left(I_{0+}^{1-\alpha}u\right)(t)\,dt\geq\int_0^T t^{\delta}|u(t)|^m\varphi(t)\,dt. \tag{3.5}$$

Note that:

$$-\int_0^T\varphi'(t)\left(I_{0+}^{1-\alpha}u\right)(t)\,dt=-\frac{1}{\Gamma(1-\alpha)}\int_0^T\varphi'(t)\left(\int_0^t\frac{u(s)}{(t-s)^{\alpha}}ds\right)dt$$
$$\leq\frac{1}{\Gamma(1-\alpha)}\int_0^T|\varphi'(t)|\left(\int_0^t\frac{|u(s)|}{(t-s)^{\alpha}}ds\right)dt.$$

Since $\varphi(t)$ is nonincreasing, we have $\varphi(s)\geq\varphi(t)$ for all $t\geq s$, and:

$$\frac{1}{\varphi(s)^{1/m}}\leq\frac{1}{\varphi(t)^{1/m}},\quad 0\leq s\leq t<T,\quad m>1.$$

Also, we have $\varphi'(t)=0$ for $t\in[0,T/2]$. Therefore:

$$-\int_0^T\varphi'(t)\left(I_{0+}^{1-\alpha}u\right)(t)\,dt\leq\frac{1}{\Gamma(1-\alpha)}\int_0^T|\varphi'(t)|\left(\int_0^t\frac{|u(s)|}{(t-s)^{\alpha}}\frac{1}{\varphi(s)^{1/m}}\varphi(s)^{1/m}ds\right)dt. \tag{2.4}$$

$$\leq\frac{1}{\Gamma(1-\alpha)}\int_0^T|\varphi'(t)|\varphi(t)^{1/m}\left(\int_0^t\frac{|u(s)|}{(t-s)^{\alpha}\varphi(s)^{1/m}}ds\right)dt \tag{2.5}$$

$$\leq\frac{1}{\Gamma(1-\alpha)}\int_{T/2}^T|\varphi'(t)|\varphi(t)^{1/m}\left(\int_0^t\frac{|u(s)|}{(t-s)^{\alpha}\varphi(s)^{1/m}}ds\right)dt. \tag{2.6}$$

Hence,

$$-\int_0^T\varphi'(t)(I_{0+}^{1-\alpha}u)(t)\,dt\leq\int_{T/2}^T|\varphi'(t)|\varphi(t)^{1/m}\left(I_{0+}^{1-\alpha}(\varphi^{1/m}|u|)\right)(t)dt.$$

By Lemma 7

$$-\int_0^T\varphi'(t)(I_{0+}^{1-\alpha}u)(t)\,dt\leq\int_{T/2}^T\left(I_{T-}^{1-\alpha}\left(\frac{|\varphi'|}{\varphi^{1/m}}\right)\right)(t)\varphi(t)^{1/m}|u(t)|\,dt. \tag{3.6}$$

(Note that we may assume $|\varphi'(t)|\varphi(t)^{-1/m}$ is summable even though $\varphi(t)\to 0$ as $t\to T$; otherwise, we consider $\varphi_\lambda(t)$ with a sufficiently large exponent λ.)

Next, we multiply inside the integral on the right-hand side of (8) by $t^{\delta/m}t^{-\delta/m}$:

$$-\int_0^T\varphi'(t)(I_{0+}^{1-\alpha}u)(t)\,dt\leq\int_{T/2}^T\left(I_{T-}^{1-\alpha}\left(\frac{|\varphi'|}{\varphi^{1/m}}\right)\right)(t)\varphi(t)^{1/m}t^{\delta/m}t^{-\delta/m}|u(t)|\,dt.$$

For $-\alpha < \delta < 0$, we have $t^{-\delta/m} < T^{-\delta/m}$ (since $t < T$), and for $\delta > 0$, we have

$$t^{-\delta/m} < 2^{\delta/m}T^{-\delta/m} \quad \text{(because } T/2 < t\text{)},$$

i.e.,

$$t^{-\delta/m} < \max\{1, 2^{\delta/m}\}T^{-\delta/m}.$$

Therefore,

$$-\int_0^T \varphi'(t)(I_{0+}^{1-\alpha}u)(t)\,dt \le \max\{1,2^{\delta/m}\}T^{-\delta/m} \times \int_{T/2}^T \left(I_{T-}^{1-\alpha}\left(\frac{|\varphi'|}{\varphi^{1/m}}\right)\right)(t)t^{\delta/m}\varphi(t)^{1/m}|u(t)|\,dt. \tag{3.7}$$

A simple application of Young's inequality (Theorem 3) with m and m' such that $\frac{1}{m}+\frac{1}{m'} = 1$ gives:

$$\begin{aligned}-\int_0^T \varphi'(t)(I_{0+}^{1-\alpha}u)(t)\,dt &\le \frac{1}{m}\int_{T/2}^T t^{\delta}\varphi(t)|u(t)|^m\,dt \\ &\quad + \frac{(\max\{1,2^{\delta/m}\})^{m'}}{m'}T^{-\delta m'/m}\int_{T/2}^T \left(I_{T-}^{1-\alpha}\left(\frac{|\varphi'|}{\varphi^{1/m}}\right)\right)^{m'}(t)\,dt \\ &\le \frac{1}{m}\int_0^T t^{\delta}\varphi(t)|u(t)|^m\,dt \\ &\quad + \frac{(\max\{1,2^{\delta/m}\})^{m'}}{m'}T^{-\delta m'/m}\int_{T/2}^T \left(I_{T-}^{1-\alpha}\left(\frac{|\varphi'|}{\varphi^{1/m}}\right)\right)^{m'}(t)\,dt.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T \varphi'(t)(I_{0+}^{1-\alpha}u)(t)\,dt &\ge -\frac{1}{m}\int_0^T t^{\delta}\varphi(t)|u(t)|^m\,dt \\ &\quad - \frac{(\max\{1,2^{\delta/m}\})^{m'}}{m'}T^{-\delta m'/m}\int_{T/2}^T \left(I_{T-}^{1-\alpha}\left(\frac{|\varphi'|}{\varphi^{1/m}}\right)\right)^{m'}(t)\,dt.\end{aligned} \tag{3.8}$$

Clearly, from (3.5) and (3.8), we conclude:

$$-b\left(I_{T-}^{\beta(1-\alpha)}\varphi\right)(0) + \frac{(\max\{1,2^{\delta/m}\})^{m'}}{m'}T^{-\delta m'/m}\int_{T/2}^T \left(I_{T-}^{1-\alpha}\left(\frac{|\varphi'|}{\varphi^{1/m}}\right)\right)^{m'}(t)\,dt.$$

We have:

$$\geq \left(1-\frac{1}{m}\right)\int_0^T t^{\delta}|u(t)|^m \varphi(t)\,dt,$$

or, since $b>0$,

$$\frac{1}{m'}\int_0^T t^{\delta}|u(t)|^m \varphi(t)\,dt \leq \left(\max\left\{1,\frac{2\delta}{m}\right\}\right)^{m'}\frac{1}{m'}T^{-\delta m'/m}\int_{T/2}^T \left(I_{T^-}^{1-\alpha}\left(\frac{|\varphi'|}{\varphi^{1/m}}\right)\right)^{m'}(t)\,dt.$$

Therefore, by Definition 7, we obtain:

$$\int_0^T t^{\delta}|u(t)|^m \varphi(t)\,dt \leq \left(\max\left\{1,\frac{2\delta}{m}\right\}\right)^{m'} T^{-\delta m'/m} \times \int_{T/2}^T \left(\frac{1}{\Gamma(1-\alpha)}\int_T^t (s-t)^{-\alpha}\frac{|\varphi'(s)|}{\varphi(s)^{1/m}}\,ds\right)^{m'} dt.$$

Letting $\sigma=\frac{t}{T}$, we get:

$$\int_0^T t^{\delta}|u(t)|^m \varphi(t)\,dt \leq \left(\max\left\{1,\frac{2\delta}{m}\right\}\right)^{m'} T^{-\delta m'/m} \times \int_{1/2}^1 \left(\frac{1}{\Gamma(1-\alpha)}\int_T^{\sigma T} (s-\sigma T)^{-\alpha}\frac{|\varphi'(s)|}{\varphi(s)^{1/m}}\,ds\right)^{m'} T\,d\sigma.$$

Making another change of variable $s=rT$, we have:

$$\int_0^T t^{\delta}|u(t)|^m \varphi(t)\,dt \leq \left(\max\left\{1,\frac{2\delta}{m}\right\}\right)^{m'} T^{-\delta m'/m} \times \int_{1/2}^1 \left(\frac{1}{\Gamma(1-\alpha)}\int_\sigma^1 (rT-\sigma T)^{-\alpha}\frac{|\varphi'(r)|}{\varphi(r)^{1/m}}T\,dr\right)^{m'} T\,d\sigma,$$

which gives:

$$\int_0^T t^{\delta}|u(t)|^m \varphi(t)\,dt \leq \left(\max\left\{1,\frac{2\delta}{m}\right\}\right)^{m'} \frac{T^{1-\alpha m'-\delta m'/m}}{\Gamma^{m'}(1-\alpha)} \times \int_{1/2}^1 \left(\int_\sigma^1 (r-\sigma)^{-\alpha}\frac{|\varphi'(r)|}{\varphi(r)^{1/m}}\,dr\right)^{m'} d\sigma.$$

It is clear that we may assume the integral term on the right-hand side is bounded, i.e.,

$$\int_{1/2}^{1}\left(\int_{\sigma}^{1}(r-\sigma)^{-\alpha}\frac{|\varphi'(r)|}{\varphi(r)^{1/m}}\,dr\right)^{m'}d\sigma \le K_1,$$

for some positive constant K_1. Otherwise, we consider $\varphi_\lambda(r)$ with a sufficiently large λ.

Hence,

$$\int_0^T t^{\delta}|u(t)|^m\varphi(t)\,dt \le K_2 T^{1-\alpha m'-\delta m'/m},$$

where

$$K_2 := \left(\max\left\{1,\frac{2\delta}{m}\right\}\right)^{m'}\Gamma^{-m'}(1-\alpha)K_1.$$

If $m < \frac{\delta+1}{1-\alpha}$, then:

$$1-\alpha m'-\frac{\delta m'}{m} < 0,$$

and consequently $T^{1-\alpha m'-\delta m'/m} \to 0$ as $T \to \infty$. Therefore, from the previous inequality:

$$\lim_{T\to\infty}\int_0^T t^{\delta}|u(t)|^m\varphi(t)\,dt = 0.$$

This is a contradiction since the solution is supposed to be nontrivial.

In the case $m = \frac{\delta+1}{1-\alpha}$, we have

$$1-\alpha m'-\frac{\delta m'}{m} = 0,$$

and relation (3.10) ensures that

$$\lim_{T\to\infty}\int_0^T t^{\delta}|u(t)|^m\varphi(t)\,dt \le K_2. \tag{3.11}$$

Moreover, it is clear that

$$\int_{T/2}^{T}\left(I_{T^-}^{1-\alpha}\left(\frac{|\varphi'|}{\varphi^{1/m}}\right)(t)\right)t^{\delta/m}\varphi(t)^{1/m}|u(t)|\,dt \le$$
$$\left[\int_{T/2}^{T}\left(I_{T^-}^{1-\alpha}\left(\frac{|\varphi'|}{\varphi^{1/m}}\right)(t)\right)^{m'}dt\right]^{1/m'}\left[\int_{T/2}^{T}t^{\delta}\varphi(t)|u(t)|^m\,dt\right]^{1/m}. \tag{2.7}$$

This inequality, together with relations (3.5) and (3.7), implies that

$$\int_0^T t^\delta \varphi(t)|u(t)|^m \, dt \leq K_3 \left[\int_{T/2}^T t^\delta \varphi(t)|u(t)|^m \, dt \right]^{1/m},$$

for some positive constant K_3, with

$$\lim_{T\to\infty} \int_{T/2}^T t^\delta \varphi(t)|u(t)|^m \, dt = 0,$$

due to the convergence of the integral in (3.11). This leads again to a contradiction. The proof is complete.

□

CHAPTER 3 NON-EXISTENCE RESULTS FOR A NONLINEAR FRACTIONAL SYSTEM OF DIFFERENTIAL PROBLEMS

3.1 Introduction

This chapter devoted to the study of the nonexistence of nontrivial global solutions for a system of fractional differential equations with a nonlinear source term and two fractional derivatives of different orders. More precisely, we consider the following system:

$$\begin{cases} D_0^{\alpha_1} u_1(\tau) = f_1\left[\tau, D_0^{\rho_1} u_1(\tau), D_0^{\rho_2} u_2(\tau)\right], \\ D_0^{\alpha_2} u_2(\tau) = f_2\left[\tau, D_0^{\rho_1} u_1(\tau), D_0^{\rho_2} u_2(\tau)\right], \end{cases} \quad \tau > 0 \tag{1}$$

where $0 < \rho_1, \rho_2 < \min\{\alpha_1, \alpha_2\}$, and D_0^{α} denotes the Riemann–Liouville fractional derivative of order α.

Furati and Kirane [2] studied the following system of fractional differential equations:

$$\begin{cases} u'(\tau) + {}^C D_0^{\rho_1} u(\tau) = |v(\tau)|^q, & \tau > 0, \quad 0 < \rho_1 < 1, \quad q > 1, \\ v'(\tau) + {}^C D_0^{\rho_2} v(\tau) = |u(\tau)|^p, & \tau > 0, \quad 0 < \rho_2 < 1, \quad p > 1, \\ u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \end{cases} \tag{2}$$

where ${}^C D_0^{\kappa}$ denotes the Caputo fractional derivative of order $\kappa > 0$.

It turns out that if $u_0 > 0$ and $v_0 > 0$ and either of the following conditions holds:

$$1 - \frac{1}{pq} < \frac{\rho_2 + \rho_1}{p} \quad \text{or} \quad 1 - \frac{1}{pq} < \frac{\rho_1 + \rho_2}{q},$$

then the system (2) admits no global solutions.

In 2021, Kassim and Tatar [12] studied the following problem:

$$D_a^{\alpha_1} u(\tau) + D_a^{\rho_1} u(\tau) \geq \left(\ln \frac{\tau}{a}\right)^{\gamma_1} |u(\tau)|^q, \quad \tau > a > 0, \quad 0 < \rho_1 < \alpha_1 < 1, \quad q > 1, \tag{3}$$

with the initial condition

$$I_a^{1-\alpha_1} u(\tau)\big|_{\tau=a} = u_0,$$

where D_a^{α} denotes the Riemann–Liouville fractional derivative of order α, and $I_a^{1-\alpha_1}$ denotes the corresponding fractional integral.

It has been shown that if $\gamma_1 > -q$, then problem (3) admits no global solutions when $u_0 \geq 0$.

The subject of this chapter is to prove that there is no nontrivial solution under appropriate assumptions on the parameters $p, q, \rho_i, \alpha_i, \gamma_i$, for $i = 1, 2$, and on the initial conditions in a suitable function space, which will be precisely defined.[14] [18] There are many findings regarding the existence of solutions to various classes of fractional differential equations, [9][10][3][1] , [16][11]. Concerning the problem of no solutions to fractional differential equations, we refer to [8], [7]

3.2 Nonexistence Result

[?]

We begin by establishing several lemmas that will be instrumental in the demonstration of the nonexistence Result of problem . The proof is mainly based on suitable manipulations of fractional integrals and related inequalities, fractional derivatives, and the test function method developed by Mitidieri and Pohozaev [17].

Definition 30. *[13] The space of absolutely continuous functions on the interval $[a,\chi]$ is denoted by $\mathrm{AC}[a,\chi]$. The space $\mathrm{AC}^n[a,\chi]$, for $n \in \mathbb{N}$, is defined by*

$$\mathrm{AC}^n[a,\chi] = \left\{ f : [a,\chi] \to \mathbb{R} \,;\, f^{(n-1)} \in \mathrm{AC}[a,\chi] \right\}.$$

Definition 31. *[13] We introduce the following function spaces:*

$$C^{\gamma}[a,\chi] = \{ f : (a,\chi] \to \mathbb{R} \mid (\tau - a)^{\gamma} f(\tau) \in C[a,\chi] \}, \quad 0 < \gamma < 1,$$

$$C^0[a,\chi] = C[a,\chi],$$

$$C^{\alpha}[a,\chi] = \left\{ f \in \mathrm{AC}^n[a,\chi] \,\middle|\, {}^{C}D_a^{\alpha} f \in C[a,\chi] \right\}, \quad n = -\lfloor -\alpha \rfloor,$$

and

$$C^{\alpha}_{n-\alpha}[a,\chi] = \left\{ f \in C^{n-\alpha}[a,\chi] \,\middle|\, D_a^{\alpha} f \in C^{n-\alpha}[a,\chi] \right\}, \quad n = -\lfloor -\alpha \rfloor,$$

Definition 32. *[13] The left-sided and right-sided Riemann–Liouville fractional integrals (RLFIs) of order $\alpha > 0$ are defined respectively by:*

$$I_a^\alpha \varphi(\tau) := \frac{1}{\Gamma(\alpha)} \int_a^\tau \frac{\varphi(s)}{(\tau - s)^{1-\alpha}} ds, \quad \tau > a, \ \alpha > 0,$$

$$I_\chi^\alpha \varphi(\tau) := \frac{1}{\Gamma(\alpha)} \int_\tau^\chi \frac{\varphi(s)}{(s - \tau)^{1-\alpha}} ds, \quad \tau < \chi, \ \alpha > 0,$$

provided that the right-hand sides exist.

When $\alpha = 0$, we define:

$$I_a^0 \varphi = I_\chi^0 \varphi = \varphi.$$

Definition 33. *[13] The left-sided and right-sided Riemann–Liouville fractional derivatives (RLFDs) of order $\alpha > 0$ are defined by*

$$D_a^\alpha \varphi(\tau) = \left(\frac{d}{d\tau}\right)^n I_a^{n-\alpha} \varphi(\tau), \quad \tau > a,$$

$$D_\chi^\alpha \varphi(\tau) = \left(-\frac{d}{d\tau}\right)^n I_\chi^{n-\alpha} \varphi(\tau), \quad \tau < \chi,$$

where $n = -\lfloor -\alpha \rfloor$.

Definition 34. *[13] The left-sided and right-sided Caputo fractional derivatives (CFDs) of order $\alpha \geq 0$ are defined by*

$${}^C D_a^\alpha \varphi(\tau) = D_a^\alpha \left[\varphi(\tau) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(a)}{k!} (\tau - a)^k\right], \quad \tau > a,$$

$${}^C D_\chi^\alpha \varphi(\tau) = D_\chi^\alpha \left[\varphi(\tau) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(\chi)}{k!} (\chi - \tau)^k\right], \quad \tau < \chi,$$

where $n = -\lfloor -\alpha \rfloor$.

Definition 35. *[13] Let $\alpha > 0$ and $n = -\lfloor -\alpha \rfloor$. If $\varphi \in AC^n[a, \chi]$, then the Caputo fractional derivatives ${}^C D_a^\alpha \varphi$ and ${}^C D_\chi^\alpha \varphi$ exist almost everywhere on $[a, \chi]$, and are represented by*

$${}^C D_a^\alpha \varphi = I_a^{n-\alpha} D^n \varphi, \quad {}^C D_\chi^\alpha \varphi = (-1)^n I_\chi^{n-\alpha} D^n \varphi.$$

Property 7. *[13] Let $\alpha \geq 0$ and $\rho > 0$. Then the following formulas hold:*

$$I_a^\alpha (\tau - a)^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho + \alpha)} (\tau - a)^{\rho+\alpha-1}, \quad I_\chi^\alpha (\chi - \tau)^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho + \alpha)} (\chi - \tau)^{\rho+\alpha-1},$$

$$D_a^\alpha (\tau - a)^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho - \alpha)} (\tau - a)^{\rho-\alpha-1}, \quad D_\chi^\alpha (\chi - \tau)^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho - \alpha)} (\chi - \tau)^{\rho-\alpha-1},$$

$${}^C D_a^\alpha (\tau - a)^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho - \alpha)} (\tau - a)^{\rho-\alpha-1}, \quad {}^C D_\chi^\alpha (\chi - \tau)^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho - \alpha)} (\chi - \tau)^{\rho-\alpha-1}.$$

Lemma 36. *[22] Let $\alpha > 0$, $m \geq 1$, $n \geq 1$, and suppose that*

$$\frac{1}{m}+\frac{1}{n} \leq 1+\alpha, \quad \text{with } m \neq 1 \text{ and } n \neq 1 \text{ when } \frac{1}{m}+\frac{1}{n} = 1+\alpha.$$

If $\varphi \in L^m(a,\chi)$ and $\psi \in L^n(a,\chi)$, then the fractional integration by parts formula holds:

$$\int_a^\chi \varphi(\tau)\,(I_a^\alpha \psi)\,(\tau)\,d\tau = \int_a^\chi \psi(\tau)\left(I_\chi^\alpha \varphi\right)(\tau)\,d\tau.$$

Lemma 37. *[19] Let $\alpha > 0$ and $n = -\lfloor -\alpha \rfloor$. If $f, I_\chi^{n-\alpha} g \in AC^n[a,\chi]$, then*

$$\int_a^\chi g(\tau)\,{}^C D_a^\alpha f(\tau)\,d\tau = \int_a^\chi f(\tau)\,D_\chi^\alpha g(\tau)\,d\tau + \sum_{i=0}^{n-1}\left[f^{(i)}(\tau)\,D_\chi^{\alpha-i-1} g(\tau)\right]_{\tau=a}^{\tau=\chi}.$$

Lemma 38. *[19] Let $\alpha > 0$ and $n = -\lfloor -\alpha \rfloor$. If $f, I_a^{n-\alpha} g \in AC^n[a,\chi]$, then*

$$\int_a^\chi f(\tau)\,D_a^\alpha g(\tau)\,d\tau = \int_a^\chi g(\tau)\,{}^C D_\chi^\alpha f(\tau)\,d\tau - \sum_{i=0}^{n-1}(-1)^{n+i}\left[D_a^{\alpha+i-n} g(\tau)\cdot D^{n-1-i} f(\tau)\right]_{\tau=a}^{\tau=\chi},$$

or equivalently,

$$\int_a^\chi f(\tau)\,D_a^\alpha g(\tau)\,d\tau = \int_a^\chi g(\tau)\,{}^C D_\chi^\alpha f(\tau)\,d\tau + \sum_{i=0}^{n-1}(-1)^i\left[f^{(i)}(\tau)\,D_a^{\alpha-i-1} g(\tau)\right]_{\tau=a}^{\tau=\chi}.$$

Lemma 39. *[4] [15] For any positive real numbers ε, φ and real number λ, we have:*

- *If $0 \leq \lambda \leq 1$, then*

$$2^{\lambda-1}(\varepsilon^\lambda + \varphi^\lambda) \leq (\varepsilon+\varphi)^\lambda \leq \varepsilon^\lambda + \varphi^\lambda.$$

- *If $\lambda \geq 1$, then*

$$\varepsilon^\lambda + \varphi^\lambda \leq (\varepsilon+\varphi)^\lambda \leq 2^{\lambda-1}(\varepsilon^\lambda + \varphi^\lambda).$$

Lemma 40. *If $\varphi \in C[a,\chi]$, then*

$$I_a^\alpha \varphi(a) = \lim_{\tau \to a^+} I_a^\alpha \varphi(\tau) = 0, \quad \text{for } \alpha > 0.$$

Proof. If $\varphi \in C[a,\chi]$, then there exists $A > 0$ such that $|\varphi(\tau)| \leq A$. Thus,

$$|I_a^\alpha \varphi(\tau)| \leq \frac{1}{\Gamma(\alpha)}\int_a^\tau (\tau-s)^{\alpha-1}|\varphi(s)|\,ds \leq \frac{A}{\Gamma(\alpha)}\int_a^\tau (\tau-s)^{\alpha-1}\,ds.$$

We compute:

$$\int_a^\tau (\tau-s)^{\alpha-1}\,ds = \left[-\frac{(\tau-s)^\alpha}{\alpha}\right]_{s=a}^{s=\tau} = \frac{(\tau-a)^\alpha}{\alpha}.$$

Therefore,

$$|I_a^\alpha \varphi(\tau)| \le \frac{A}{\Gamma(\alpha)}\cdot\frac{(\tau-a)^\alpha}{\alpha}.$$

Since $\alpha > 0$, we conclude that

$$\lim_{\tau\to a^+} I_a^\alpha \varphi(\tau) = 0.$$

□

We consider the test function

$$\phi(\tau) = \begin{cases}(\chi-\kappa)(\chi-\tau)^\kappa, & 0\le\tau\le\chi, \quad \kappa>0,\\ 0, & \tau>\chi.\end{cases}$$

In the next lemma we prove our first estimation.

Lemma 41. *Let* $\alpha \ge 0$ *and* $\phi(\tau)$ *be the test function defined as in . Then*

$$I_\chi^\alpha \phi(\tau) = \frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa+\alpha)}\chi^{-\kappa}(\chi-\tau)^{\kappa+\alpha},$$

and

$${}^C D_\chi^\alpha \phi(\tau) = D_\chi^\alpha \phi(\tau) = \frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa-\alpha)}\chi^{-\kappa}(\chi-\tau)^{\kappa-\alpha}.$$

Proof. The result follows directly from Property 7. □

Lemma 42. *Let* $\alpha > 0$*,* $n = -\lfloor -\alpha \rfloor$*, and let* ϕ *be the function defined as in with* $\kappa > \max\{0, \alpha-1\}$*. If* $f \in AC^n[0,\chi]$*, where* $\chi > 0$*, then*

$$\int_0^\chi \phi(\tau)\,{}^C D_0^\alpha f(\tau)\,d\tau = \int_0^\chi f(\tau)\,D_\chi^\alpha \phi(\tau)\,d\tau - \sum_{i=0}^{n-1}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\alpha+i+2)}\chi^{-\alpha+i+1} f^{(i)}(0).$$

Proof. From Lemma 41, we know:

$$D_\chi^{\alpha-i-1}\phi(\tau) = \frac{\Gamma(1+\kappa)}{\Gamma(2+\kappa-\alpha+i)}\chi^{-\kappa}(\chi-\tau)^{\kappa-\alpha+i+1}.$$

Then evaluating at $\tau = 0$, we get:

$$D_\chi^{\alpha-i-1}\phi(0) = \frac{\Gamma(1+\kappa)}{\Gamma(2+\kappa-\alpha+i)}\chi^{-\alpha+i+1}, \quad \text{and} \quad D_\chi^{\alpha-i-1}\phi(\chi) = 0.$$

The result follows by applying Lemma 37. □

Lemma 43. *Let* $\alpha > 0$, $n = -\lfloor -\alpha \rfloor$, *and let* ϕ *be as in* (3.1) *with* $\kappa > n-1$. *If* $I_0^{n-\alpha} f \in AC^n[0,\chi]$, *with* $\chi > 0$, *then*

$$\int_0^{\chi} \phi(\tau)\, D_0^{\alpha} f(\tau)\, d\tau = \int_0^{\chi} f(\tau)\, {}^C D_{\chi}^{\alpha} \phi(\tau)\, d\tau - \sum_{i=0}^{n-1} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-i)} \chi^{-i} D_0^{\alpha-i-1} f(0).$$

Proof. We compute:

$$D^{(i)}\phi(\tau) = (-1)^i \kappa(\kappa-1)\cdots(\kappa-i+1)\chi^{-\kappa}(\chi-\tau)^{\kappa-i} = (-1)^i \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-i)} \chi^{-\kappa}(\chi-\tau)^{\kappa-i}.$$

Hence,

$$D^{(i)}\phi(0) = (-1)^i \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-i)} \chi^{-i}, \quad \text{and} \quad D_{\chi}^{(i)}\phi(\chi) = 0.$$

The result follows from Lemma 38. □

$$\phi(\tau) = \begin{cases} (\chi-\kappa)(\chi-\tau)^{\kappa}, & 0 \le \tau \le \chi, \quad \kappa > 0, \\ 0, & \tau > \chi. \end{cases} \tag{3.1}$$

Lemma 44. *Let* ϕ *be as in* (3.1) *with* $\kappa > \max\{p(\rho-\alpha)-1, \rho-1\}$, *where* $\rho, \alpha \ge 0$ *and* $p > 1$. *Then*

$$\int_0^{\chi} \tau^{\gamma(1-p)} \phi^{1-p}(\tau) \left[I_{\chi}^{\alpha} D_{\chi}^{\rho} \phi(\tau) \right]^p d\tau = C_{\kappa,\alpha,\rho}^{\gamma,p} \chi^{\gamma(1-p)+p(\alpha-\rho)+1}, \quad \textit{where } \gamma(1-p)+1 > 0,$$

and

$$C_{\kappa,\alpha,\rho}^{\gamma,p} = \left[\frac{\Gamma(\kappa+1)}{\Gamma(\alpha+\kappa-\rho+1)} \right]^p \frac{\Gamma(p(\alpha-\rho)+\kappa+1)\Gamma(\gamma(1-p)+1)}{\Gamma(\gamma(1-p)+p(\alpha-\rho)+\kappa+2)}.$$

Lemma 45. *Let* ϕ *be as in* (3.1) *with* $\kappa > p\rho - 1$, $\rho > 0$, $\gamma(1-p)+1 > 0$, *and* $p > 1$. *Then*

$$\int_0^{\chi} \tau^{\gamma(1-p)} \phi^{1-p}(\tau) \left[D_{\chi}^{\rho} \phi(\tau) \right]^p d\tau = C_{\kappa,\rho}^{\gamma,p} \chi^{\gamma(1-p)-p\rho+1},$$

where

$$C_{\kappa,\rho}^{\gamma,p} = \left[\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\rho+1)} \right]^p \frac{\Gamma(\kappa-p\rho+1)\Gamma(\gamma(1-p)+1)}{\Gamma(\gamma(1-p)+\kappa-p\rho+2)}.$$

Lemma 46. *Let* $\alpha > 0$, $n = -\lfloor -\alpha \rfloor$, *and let* $\phi(\tau)$ *be as in* (3.1) *with* $\kappa > n-1$. *If* $I^{n-\alpha} f \in AC^n[a,\chi]$, *then*

$$\int_0^{\chi} \phi(\tau) D_a^{\alpha} f(\tau)\, d\tau = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\alpha)} \chi^{-\kappa} \int_0^{\chi} (\chi-\tau)^{\kappa-\alpha} f(\tau)\, d\tau - \sum_{i=0}^{n-1} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-i)} \chi^{-i} D_0^{\alpha-i-1} f(0).$$

Proof. It follows from Property 7 and Lemma 43 . □

Lemma 47. *Let* $\alpha > 0$*,* $n = -\lfloor -\alpha \rfloor$*, and* $\phi(\tau)$ *be as in* (3.1) *with* $\kappa > \max\{0, \alpha - 1\}$*. If* $f \in AC^n[a, \chi]$*, then*

$$\int_0^\chi \phi(\tau)\, {}^C D_0^\alpha f(\tau)\, d\tau = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\alpha)} \chi^{-\kappa} \int_0^\chi (\chi-\tau)^{\kappa-\alpha} f(\tau)\, d\tau - \sum_{i=0}^{n-1} \frac{\Gamma(1+\kappa)}{\Gamma(\kappa-\alpha+i+2)} \chi^{-\alpha+i+1} f^{(i)}(0).$$

Proof. It follows from Property 7 and Lemma 42 . □

Lemma 48. *Let* $\alpha, \rho \geq 0$ *and let* $\phi(\tau)$ *be as in* (3.1) *with* $\kappa + 1 > \rho$*. Then*

$$I_\chi^\alpha D_\chi^\rho \phi(\tau) = \frac{\Gamma(\kappa+1)}{\Gamma(1+\kappa+\alpha-\rho)} \chi^{-\kappa} (\chi-\tau)^{\alpha+\kappa-\rho}.$$

Proof of Lemma 48. By using Lemma 41 , we have

$$D_\chi^\rho \phi(\tau) = \frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa-\rho)} \chi^{-\kappa} (\chi-\tau)^{\kappa-\rho}.$$

By virtue of Property 7, we get

$$I_\chi^\alpha D_\chi^\rho \phi(\tau) = \frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa-\rho)} \chi^{-\kappa} I_\chi^\alpha (\chi-\tau)^{\kappa-\rho}.$$

Since

$$I_\chi^\alpha (\chi-\tau)^{\kappa-\rho} = \frac{\Gamma(\kappa-\rho+1)}{\Gamma(\kappa+\alpha-\rho+1)} (\chi-\tau)^{\kappa+\alpha-\rho},$$

we conclude that

$$I_\chi^\alpha D_\chi^\rho \phi(\tau) = \frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa+\alpha-\rho)} \chi^{-\kappa} (\chi-\tau)^{\alpha+\kappa-\rho}.$$

□

Proof. By using Lemma 41, we have

$$D_\chi^\rho \phi(\tau) = \frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa-\rho)} \chi^{-\kappa} (\chi-\tau)^{\kappa-\rho}.$$

By virtue of Property 7, we get

$$I_\chi^\alpha D_\chi^\rho \phi(\tau) = \frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa-\rho)} \chi^{-\kappa} I_\chi^\alpha (\chi-\tau)^{\kappa-\rho}.$$

Using the fractional integral formula, this becomes

$$I_{\chi}^{\alpha} D_{\chi}^{\rho} \phi(\tau) = \frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa+\alpha-\rho)} \chi^{-\kappa} (\chi-\tau)^{\alpha+\kappa-\rho}.$$

□

Lemma 49. *Let* $\phi(\tau)$ *be as in (7) with* $\kappa > q\rho - 1$*,* $\rho > 0$*,* $\gamma(1-q)+1 > 0$ *and* $q > 1$*. Then*

$$\int_0^{\chi} \tau^{\gamma(1-q)} (\chi-\tau)^{q(\kappa-\rho)} \phi^{1-q}(\tau)\, d\tau = K_{\kappa,\rho}^{\gamma,q} \chi^{\gamma+1+q(\kappa-\gamma-\rho)},$$

where

$$K_{\kappa,\rho}^{\gamma,q} = \frac{\Gamma(\kappa+1-\rho q)\,\Gamma(\gamma(1-q)+1)}{\Gamma(\gamma(1-q)+\kappa-\rho q+2)}.$$

Proof. We have from (7):

$$(\chi-\tau)^{q(\kappa-\rho)} \phi^{1-q}(\tau) = (\chi-\tau)^{q(\kappa-\rho)} \left[\chi^{-\kappa}(\chi-\tau)^{\kappa}\right]^{1-q} = \chi^{-\kappa(1-q)} (\chi-\tau)^{\kappa-\rho q}.$$

Then

$$\int_0^{\chi} \tau^{\gamma(1-q)} (\chi-\tau)^{q(\kappa-\rho)} \phi^{1-q}(\tau)\, d\tau = \chi^{-\kappa(1-q)} \int_0^{\chi} \tau^{\gamma(1-q)} (\chi-\tau)^{\kappa-\rho q}\, d\tau.$$

Make the substitution $\tau = \zeta\chi$, so $d\tau = \chi\, d\zeta$, and the limits become from 0 to 1. Then:

$$= \chi^{-\kappa(1-q)} \chi^{\gamma(1-q)+\kappa-\rho q+1} \int_0^1 \zeta^{\gamma(1-q)} (1-\zeta)^{\kappa-\rho q}\, d\zeta.$$

So we have:

$$= \chi^{\gamma+1+q(\kappa-\gamma-\rho)} \cdot \frac{\Gamma(\kappa+1-\rho q)\Gamma(\gamma(1-q)+1)}{\Gamma(\gamma(1-q)+\kappa-\rho q+2)}.$$

□

Lemma 50. *Let* $\phi(\tau)$ *be as in (7) with* $\kappa > q\rho - 1$*,* $\rho > 0$*,* $\gamma(1-q) > -1$ *and* $q > 1$*. Then*

$$\int_0^{\chi} \tau^{\gamma(1-q)} \phi^{1-q}(\tau) \left[D_{\chi}^{\rho} \phi(\tau)\right]^q d\tau = C_{\kappa,\rho}^{\gamma,q} \chi^{\gamma(1-q)-\rho q+1},$$

where

$$C_{\kappa,\rho}^{\gamma,q} = \frac{\Gamma(\kappa+1-\rho q)\,\Gamma(\gamma(1-q)+1)}{\Gamma(2+\gamma(1-q)+\kappa-\rho q)} \left[\frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa-\rho)}\right]^q.$$

3.3 Non existence Result for the System with Riemann–Liouville Derivatives

In this section, we discuss the system 1 with Riemann–Liouville fractional derivatives (RLFD). We will prove a nonexistence result for global solutions of the system 1 under the following assumptions:

$$\begin{aligned} f_1\left[\tau, D_0^{\rho_1}u_1(\tau), D_0^{\rho_2}u_2(\tau)\right] &= \tau^{\gamma_2}|u_2(\tau)|^{p_2} - \lambda_1 D_0^{\rho_1}u_1(\tau), \quad \tau > 0, \\ f_2\left[\tau, D_0^{\rho_1}u_1(\tau), D_0^{\rho_2}u_2(\tau)\right] &= \tau^{\gamma_1}|u_1(\tau)|^{p_1} - \lambda_2 D_0^{\rho_2}u_2(\tau), \quad \tau > 0, \end{aligned}$$

for some $p_j > 1$ and $\lambda_j, \gamma_j \in \mathbb{R}$, $j = 1,2$. The problem is considered in the space $C^{\alpha_1}_{n-\alpha_1}[0,\infty) \times C^{\alpha_2}_{n-\alpha_2}[0,\infty)$, where $C^{\alpha}_{n-\alpha}[0,\infty)$ is defined as in equation (5).

That is, we consider the system:

$$\begin{cases} D_0^{\alpha_1}u_1(\tau) + \lambda_1 D_0^{\rho_1}u_1(\tau) = \tau^{\gamma_2}|u_2(\tau)|^{p_2}, & 0 < \rho_1 < \alpha_1,\ \tau > 0, \\ D_0^{\alpha_2}u_2(\tau) + \lambda_2 D_0^{\rho_2}u_2(\tau) = \tau^{\gamma_1}|u_1(\tau)|^{p_1}, & 0 < \rho_2 < \alpha_2,\ \tau > 0, \end{cases} \tag{8}$$

subject to the initial conditions:

$$D_0^{\alpha_j - k}u_j(\tau)\big|_{\tau=0} = b_{j,k} \in \mathbb{R}, \quad j = 1,2, \quad k = 1,\ldots,n = -\lfloor -\alpha_j \rfloor. \tag{9}$$

Theorem 51. *Assume that*

$$1 - \frac{1}{p_1 p_2} < \frac{\rho_1 + \rho_2}{p_1} + \frac{\gamma_1}{p_1} + \frac{\gamma_2}{p_1 p_2} \quad \text{or} \quad 1 - \frac{1}{p_1 p_2} < \frac{\rho_2 + \rho_1}{p_2} + \frac{\gamma_2}{p_2} + \frac{\gamma_1}{p_1 p_2}, \tag{10}$$

where $p_j > 1$ and $\gamma_j < p_j - 1$, $j = 1,2$. Then, the problem (8) - (9) does not admit nontrivial global solution in $C^{\alpha_1}_{n-\alpha_1}[0,\infty) \times C^{\alpha_2}_{n-\alpha_2}[0,\infty)$, provided that $b_{j,k} \geq 0$ for all $j = 1,2$, $k = 1,\ldots,n = -\lfloor -\alpha_j \rfloor$.

Proof. We argue by contradiction. Assume that (u_1,u_2) is a global solution. Let $\varphi(\tau)$ be as in equation (7), with

$$\kappa > \max\left\{n-1, \frac{p_j\alpha_j}{p_j - 1} - 1\right\}, \quad j = 1,2.$$

Multiplying the system (8) by $\varphi(\tau)$ and integrating over $(0,\chi)$, we obtain:

$$I_2 := \int_0^\chi \tau^{\gamma_2}|u_2(\tau)|^{p_2}\varphi(\tau)\,d\tau = \int_0^\chi D_0^{\alpha_1}u_1(\tau)\varphi(\tau)\,d\tau + \lambda_1\int_0^\chi D_0^{\rho_1}u_1(\tau)\varphi(\tau)\,d\tau, \tag{11}$$

$$I_1 := \int_0^\chi \tau^{\gamma_1}|u_1(\tau)|^{p_1}\varphi(\tau)\,d\tau = \int_0^\chi D_0^{\alpha_2}u_2(\tau)\varphi(\tau)\,d\tau + \lambda_2\int_0^\chi D_0^{\rho_2}u_2(\tau)\varphi(\tau)\,d\tau. \tag{12}$$

Define

$$J_j := \int_0^\chi D_0^{\alpha_j} u_j(\tau)\varphi(\tau)\,d\tau, \tag{13}$$

$$H_j := \lambda_j \int_0^\chi D_0^{\rho_j} u_j(\tau)\varphi(\tau)\,d\tau, \quad j = 1,2. \tag{14}$$

We now estimate J_j and H_j, for $j = 1,2$. From Lemma 46, we have:

$$J_j = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\alpha_j)}\chi^{-\kappa}\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} u_j(\tau)\,d\tau - \sum_{i=0}^{n-1}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-i)} b_{j,i}\chi^{-i}, \quad j = 1,2. \tag{15}$$

Now, multiply inside the integral in (15) by

$$\frac{\tau^{\gamma_j/p_j}\varphi(\tau)^{1/p_j}}{\tau^{\gamma_j/p_j}\varphi(\tau)^{1/p_j}} = 1,$$

to obtain

$$\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} u_j(\tau)\,d\tau = \int_0^\chi \tau^{\gamma_j/p_j} u_j(\tau)\varphi(\tau)^{1/p_j}(\chi-\tau)^{\kappa-\alpha_j}\tau^{-\gamma_j/p_j}\varphi(\tau)^{-1/p_j}\,d\tau.$$

Using Hölder's inequality with conjugate exponents p_j and p'_j, where $\frac{1}{p_j} + \frac{1}{p'_j} = 1$, we get:

$$\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} u_j(\tau)\,d\tau \leq \left(\int_0^\chi \tau^{\gamma_j}\varphi(\tau)|u_j(\tau)|^{p_j}\,d\tau\right)^{1/p_j}$$
$$\times \left(\int_0^\chi \tau^{-p'_j\gamma_j/p_j}\varphi(\tau)^{-p'_j/p_j}(\chi-\tau)^{(\kappa-\alpha_j)p'_j} d\tau\right)^{1/p'_j}.$$

By Lemma 49 , it follows that:

$$\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} u_j(\tau)\,d\tau \leq \left[K_{\kappa,\alpha_j}^{\gamma_j,p'_j}\right]^{1/p'_j} \chi^{\kappa+\frac{1}{p'_j}-\alpha_j+\frac{\gamma_j}{p'_j}-\frac{\gamma_j}{p_j}} I_j^{1/p_j}, \quad j = 1,2.$$

Therefore from (15) and (16), we get:

$\mathrm{J}_j \leq K_j\chi^{\frac{1}{p'_j}-\alpha_j+\frac{\gamma_j}{p'_j}-\frac{\gamma_j}{p_j}} I_j^{1/p_j} - \sum_{i=0}^{n-1}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-i)} b_{j,i}\chi^{-i}, \quad j = 1,2, (17)$

With

$$K_j = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\alpha_j)}\left[K^{\kappa,\alpha_j}_{\gamma_j,p'_j}\right]^{\frac{1}{p'_j}}, \quad j=1,2.$$

Since $b_{j,i} \geq 0$, $j=1,2$, then

$$J_j \leq K_j \chi^{\frac{1}{p'_j}-\alpha_j+\frac{\gamma_j}{p'_j}-\gamma_j} I_j^{\frac{1}{p_j}}, \quad j=1,2. \tag{18}$$

Now, we turn to H_j, $j=1,2$. First, by Lemma 40, we see that

$$D_0^{\rho_j-n}u_j(0) = I_0^{n-\rho_j}u_j(0) = I_0^{\alpha_j-\rho_j} I_0^{n-\alpha_j}u_j(0) = 0, \quad j=1,2,$$

because $I_0^{n-\alpha_j}u_j(0) \in AC^n[0,\chi]$. From Lemma 43 , we can write

$$\begin{aligned} H_j = \lambda_j \int_0^{\chi} \phi(\tau) D_0^{\rho_j} u_j(\tau)\,d\tau = \lambda_j \Bigg[\int_0^{\chi} u_j(\tau)\, {}^C D_{\chi}^{\rho_j}\phi(\tau)\,d\tau \\ - \sum_{i=0}^{n-1} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-i)}\chi^{-i} D_0^{\rho_j-i-1}u_j(0)\Bigg] \end{aligned}$$

$$= \lambda_j \int_0^{\chi} u_j(\tau)\, {}^C D_{\chi}^{\rho_j}\phi(\tau)\,d\tau, \quad j=1,2. \tag{19}$$

Again from Lemma 46 ,we can write

$$H_j = \lambda_j \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\rho_j)}\chi^{-\kappa}\int_0^{\chi}(\chi-\tau)^{\kappa-\rho_j}u_j(\tau)\,d\tau, \quad j=1,2.$$

Similarly to J_i, $i=1,2$, we have

$$H_j \leq K'_j \chi^{\frac{1}{p'_j}-\rho_j+\frac{\gamma_j}{p'_j}-\gamma_j} I_j^{\frac{1}{p_j}}, \quad j=1,2. \tag{20}$$

with

$$K'_j = |\lambda_j| \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\rho_j)}\left[K^{\kappa,\rho_j}_{\gamma_j,p'_j}\right]^{\frac{1}{p'_j}}, \quad j=1,2.$$

We use (18) and (20) to write (11) and (12) in the form:

$$I_2 \leq K''_1 \chi^{\frac{1}{p'_1}+\frac{\gamma_1}{p'_1}-\gamma_1} I_1^{\frac{1}{p_1}}(\chi-\alpha_1+\chi-\rho_1), \tag{21}$$

$$I_1 \leq K''_2 \chi^{\frac{1}{p'_2}+\frac{\gamma_2}{p'_2}-\gamma_2} I_2^{\frac{1}{p_2}}(\chi-\alpha_2+\chi-\rho_2), \tag{22}$$

with

$$K''_j = \max\{K_j, K'_j\}, \quad j=1,2.$$

Consequently, (21) and (22) become

$$I_1^{1-\frac{1}{p_1p_2}} \leq K_1''(K_2'')^{\frac{1}{p_1}}\chi^{\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1+\frac{1}{p_1p_2'}+\frac{\gamma_2}{p_1p_2'}-\frac{\gamma_2}{p_1}}(\chi-\alpha_2+\chi-\rho_2)^{\frac{1}{p_1}}(\chi-\alpha_1+\chi-\rho_1). \tag{23}$$

And

$$I_1^{1-\frac{1}{p_1p_2}} \leq K_2''(K_1'')^{\frac{1}{p_2}}\chi^{\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2+\frac{1}{p_2p_1'}+\frac{\gamma_1}{p_2p_1'}-\frac{\gamma_1}{p_2}}(\chi-\alpha_1+\chi-\rho_1)^{\frac{1}{p_2}}(\chi-\alpha_2+\chi-\rho_2). \tag{24}$$

Using Lemma 39 with $0 \leq \lambda \leq 1$, we estimate (23) and (24) as:

$$I_1^{1-\frac{1}{p_1p_2}} \leq K_1''(K_2'')^{\frac{1}{p_1}}\chi^{\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1+\frac{1}{p_1p_2'}+\frac{\gamma_2}{p_1p_2'}-\frac{\gamma_2}{p_1}}\left(\frac{\chi-\alpha_2}{p_1}+\frac{\chi-\rho_2}{p_1}\right)(\chi-\alpha_1+\chi-\rho_1),$$

and

$$I_1^{1-\frac{1}{p_1p_2}} \leq K_2''(K_1'')^{\frac{1}{p_2}}\chi^{\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2+\frac{1}{p_2p_1'}+\frac{\gamma_1}{p_2p_1'}-\frac{\gamma_1}{p_2}}\left(\frac{\chi-\alpha_1}{p_2}+\frac{\chi-\rho_1}{p_2}\right)(\chi-\alpha_2+\chi-\rho_2),$$

or in simplified notation,

$$I_1^{1-\frac{1}{p_1p_2}} \leq K_1''(K_2'')^{1/p_1}\left(\chi^{s_1}+\chi^{s_2}+\chi^{s_3}+\chi^{s_4}\right), \tag{25}$$

and

$$I_1^{1-\frac{1}{p_1p_2}} \leq K_2''(K_1'')^{1/p_2}\left(\chi^{s_5}+\chi^{s_6}+\chi^{s_7}+\chi^{s_8}\right), \tag{26}$$

where:

$$\begin{aligned}
s_1 &= -\frac{\alpha_2}{p_1}-\alpha_1+\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1+\frac{1}{p_1p_2'}+\frac{\gamma_2}{p_1p_2'}-\frac{\gamma_2}{p_1},\\
s_2 &= -\frac{\alpha_2}{p_1}-\rho_1+\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1+\frac{1}{p_1p_2'}+\frac{\gamma_2}{p_1p_2'}-\frac{\gamma_2}{p_1},\\
s_3 &= -\frac{\rho_2}{p_1}-\alpha_1+\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1+\frac{1}{p_1p_2'}+\frac{\gamma_2}{p_1p_2'}-\frac{\gamma_2}{p_1},\\
s_4 &= -\frac{\rho_2}{p_1}-\rho_1+\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1+\frac{1}{p_1p_2'}+\frac{\gamma_2}{p_1p_2'}-\frac{\gamma_2}{p_1},\\
s_5 &= -\frac{\alpha_1}{p_2}-\alpha_2+\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2+\frac{1}{p_2p_1'}+\frac{\gamma_1}{p_2p_1'}-\frac{\gamma_1}{p_2},\\
s_6 &= -\frac{\alpha_1}{p_2}-\rho_2+\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2+\frac{1}{p_2p_1'}+\frac{\gamma_1}{p_2p_1'}-\frac{\gamma_1}{p_2},\\
s_7 &= -\frac{\rho_1}{p_2}-\alpha_2+\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2+\frac{1}{p_2p_1'}+\frac{\gamma_1}{p_2p_1'}-\frac{\gamma_1}{p_2},\\
s_8 &= -\frac{\rho_1}{p_2}-\rho_2+\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2+\frac{1}{p_2p_1'}+\frac{\gamma_1}{p_2p_1'}-\frac{\gamma_1}{p_2}.
\end{aligned}$$

If $1-\frac{1}{p_1p_2}<\frac{\rho_1+\rho_2}{p_1}+\frac{\gamma_1}{p_1}+\frac{\gamma_2}{p_1p_2}$, then $s_m<0$, for $m=1,2,3,4$; or if $1-\frac{1}{p_1p_2}<\frac{\rho_2+\rho_1}{p_2}+\frac{\gamma_2}{p_2}+\frac{\gamma_1}{p_1p_2}$, then $s_m<0$, for $m=5,6,7,8$, and therefore $\chi^{s_m}\to 0$ as $\chi\to\infty$. Then,

$$\lim_{\chi\to\infty}\int_0^{\chi}\tau^{\gamma_j}\phi(\tau)|u_j(\tau)|^{p_j}d\tau=0,\quad j=1,2.$$

This means that $u_j=0$, $j=1,2$. We reach a contradiction. □

Theorem 52. *Assume that*

$$1-\frac{1}{p_1p_2}\leq\frac{\rho_1+\rho_2}{p_1}+\frac{\gamma_1}{p_1}+\frac{\gamma_2}{p_1p_2}\quad or\quad 1-\frac{1}{p_1p_2}\leq\frac{\rho_2+\rho_1}{p_2}+\frac{\gamma_2}{p_2}+\frac{\gamma_1}{p_1p_2},$$

where

$$(1-\rho_j)p_j-1<\gamma_j<p_j-1,\quad j=1,2.$$

Then, the system (8) admits no global nontrivial solutions in the space

$$C^{\alpha_1}_{n-\alpha_1}[0,\infty)\times C^{\alpha_2}_{n-\alpha_2}[0,\infty)$$

when $b_{j,0}>0$, $j=1,2$.

Proof. From the previous theorem (see equations (25) and (26)), we have:

$$I_1^{1-\frac{1}{p_1p_2}}\leq K_1''(K_2'')^{1/p_1}\left(\chi^{s_1}+\chi^{s_2}+\chi^{s_3}+\chi^{s_4}\right),$$

and

$$I_1^{1-\frac{1}{p_1p_2}}\leq K_2''(K_1'')^{1/p_2}\left(\chi^{s_5}+\chi^{s_6}+\chi^{s_7}+\chi^{s_8}\right).$$

If

$$1-\frac{1}{p_1p_2}\leq\frac{\rho_1+\rho_2}{p_1}+\frac{\gamma_1}{p_1}+\frac{\gamma_2}{p_1p_2},$$

then $s_m\leq 0$, for $m=1,2,3,4$; or if

$$1-\frac{1}{p_1p_2}\leq\frac{\rho_2+\rho_1}{p_2}+\frac{\gamma_2}{p_2}+\frac{\gamma_1}{p_1p_2},$$

then $s_m\leq 0$, for $m=5,6,7,8$. Therefore, I_1 and I_2 are bounded.

Now, using equations (11), (12), (17), and (20), we obtain:

$$I_2\leq K_1\chi^{\frac{1}{p_1'}-\alpha_1+\frac{\gamma_1}{p_1'}-\gamma_1}I_1^{1/p_1}+K_1'\chi^{\frac{1}{p_1'}-\rho_1+\frac{\gamma_1}{p_1'}-\gamma_1}I_1^{1/p_1}-\sum_{i=0}^{n-1}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa+1-i)}\chi^{-i},$$

and

$$I_1\leq K_2\chi^{\frac{1}{p_2'}-\alpha_2+\frac{\gamma_2}{p_2'}-\gamma_2}I_2^{1/p_2}+K_2'\chi^{\frac{1}{p_2'}-\rho_2+\frac{\gamma_2}{p_2'}-\gamma_2}I_2^{1/p_2}-\sum_{i=0}^{n-1}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa+1-i)}\chi^{-i}.$$

Or, more compactly:

$$I_2 + b_{1,0} \leq \max\{K_1, K_1'\}\chi^{\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1}\left(\chi^{-\alpha_1}+\chi^{-\rho_1}\right) I_1^{1/p_1} - \sum_{i=1}^{n-1}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa+1-i)}\chi^{-i},$$

$$I_1 + b_{2,0} \leq \max\{K_2, K_2'\}\chi^{\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2}\left(\chi^{-\alpha_2}+\chi^{-\rho_2}\right) I_2^{1/p_2} - \sum_{i=1}^{n-1}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa+1-i)}\chi^{-i}.$$

Since $I_j \geq 0$ for $j = 1,2$, we deduce:

$$b_{1,0} \leq \max\{K_1, K_1'\}\chi^{\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1}\left(\chi^{-\alpha_1}+\chi^{-\rho_1}\right) I_1^{1/p_1} - \sum_{i=1}^{n-1}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa+1-i)}\chi^{-i},$$

$$b_{2,0} \leq \max\{K_2, K_2'\}\chi^{\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2}\left(\chi^{-\alpha_2}+\chi^{-\rho_2}\right) I_2^{1/p_2} - \sum_{i=1}^{n-1}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa+1-i)}\chi^{-i}.$$

As $\chi \to \infty$, the right-hand sides of the above inequalities tend to zero. Thus, we obtain a contradiction:

$$0 < b_{j,0} \leq 0, \quad j = 1,2.$$

□

3.4 4 Non-Existence of Solutions in the Case of Caputo Fractional Derivatives

In this section, we consider the system

$$\begin{aligned} {}^{C}D_0^{\alpha_1} y_1(\tau) + \lambda_1 {}^{C}D_0^{\rho_1} y_1(\tau) &\geq \tau^{\gamma_2}|y_2(\tau)|^{p_2}, \quad 0<\rho_1<\alpha_1,\ \tau>0, \quad (3.2)\\ {}^{C}D_0^{\alpha_2} y_2(\tau) + \lambda_2 {}^{C}D_0^{\rho_2} y_2(\tau) &\geq \tau^{\gamma_1}|y_1(\tau)|^{p_1}, \quad 0<\rho_2<\alpha_2,\ \tau>0. \quad (27)\end{aligned}$$

with the initial conditions

$$y_j^{(k)}(0) = b_{j,k}, \quad k = 0,\ldots,n-1, \quad n = -\lceil \alpha_j \rceil, \quad b_{j,k}\in\mathbb{R}, \quad j=1,2. \tag{28}$$

where, ${}^{C}D_0^{\sigma}$ is the Caputo fractional derivative of order σ, $p_j > 1$, and $\gamma_j \in \mathbb{R}$, for $j = 1,2$.

Theorem 53. *Assume that*

$$1-\frac{1}{p_1p_2} < \frac{\rho_2}{p_1}+\frac{\rho_1+\gamma_1}{p_1}+\frac{\gamma_2}{p_1p_2} \quad \textit{or} \quad 1-\frac{1}{p_1p_2} < \frac{\rho_1}{p_2}+\frac{\rho_2+\gamma_2}{p_2}+\frac{\gamma_1}{p_1p_2},$$

where $p_j > 1$ and $\gamma_j < p_j - 1$ for $j = 1,2$. Then, Problem (27)-(28) does not admit nontrivial global solutions in the space $C^{\alpha_1}[0,\infty)\times C^{\alpha_2}[0,\infty)$, where $C^{\alpha}[0,\infty)$ is as in (4), when $b_{j,k} \geq 0$, for $j = 1,2$.

Proof. We argue by contradiction. Suppose that a global nontrivial solution $(y_1, y_2) \in C^{\alpha_1}[0,\infty) \times C^{\alpha_2}[0,\infty)$ exists for all $\tau > 0$.

Let φ be as in (7) with

$$\kappa > \max\left\{0, \alpha_j - 1, \frac{p_j \alpha_j}{p_j - 1} - 1\right\}, \quad j = 1,2.$$

Multiplying equations (27) and (28) by φ and integrating over $[0,\chi]$, we obtain:

$$\int_0^\chi \varphi(\tau)\tau^{\gamma_2}|y_2(\tau)|^{p_2}\, d\tau \leq \int_0^\chi \varphi(\tau)\, {}^C D_0^{\alpha_1} y_1(\tau)\, d\tau + \lambda_1 \int_0^\chi \varphi(\tau)\, {}^C D_0^{\rho_1} y_1(\tau)\, d\tau, \tag{29}$$

$$\int_0^\chi \varphi(\tau)\tau^{\gamma_1}|y_1(\tau)|^{p_1}\, d\tau \leq \int_0^\chi \varphi(\tau)\, {}^C D_0^{\alpha_2} y_2(\tau)\, d\tau + \lambda_2 \int_0^\chi \varphi(\tau)\, {}^C D_0^{\rho_2} y_2(\tau)\, d\tau. \tag{30}$$

Define:

$$I_j = \int_0^\chi \varphi(\tau)\tau^{\gamma_j}|y_j(\tau)|^{p_j}\, d\tau, \quad j = 1,2, \tag{31}$$

$$J_j = \int_0^\chi \varphi(\tau)\, {}^C D_0^{\alpha_j} y_j(\tau)\, d\tau, \quad j = 1,2, \tag{32}$$

$$H_j = \lambda_j \int_0^\chi \varphi(\tau)\, {}^C D_0^{\rho_j} y_j(\tau)\, d\tau, \quad j = 1,2. \tag{33}$$

By Lemma 42, we can write:

$$J_j = \int_0^\chi y_j(\tau) D_\chi^{\alpha_j} \varphi(\tau)\, d\tau - \sum_{i=0}^{n-1} \frac{\Gamma(\kappa+1) b_{j,i}}{\Gamma(\kappa - \alpha_j + i + 2)} \chi^{1+i-\alpha_j}, \tag{34}$$

$$H_j = \lambda_j \left[\int_0^\chi y_j(\tau) D_\chi^{\rho_j} \varphi(\tau)\, d\tau - \sum_{i=0}^{n'-1} \frac{\Gamma(\kappa+1) b_{j,i}}{\Gamma(\kappa - \rho_j + i + 2)} \chi^{1+i-\rho_j}\right]. \tag{35}$$

□

where $n' = -[\rho_j]$, $j = 1,2$. By Lemma 47, we obtain:

$$J_j = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\alpha_j)} \chi^{-\kappa} \int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} y_j(\tau)\, d\tau - \sum_{i=0}^{n-1} \frac{\Gamma(\kappa+1) b_{j,i}}{\Gamma(\kappa - \alpha_j + i + 2)} \chi^{1+i-\alpha_j}, \tag{36}$$

$$H_j = \lambda_j \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\rho_j)} \chi^{-\kappa} \int_0^\chi (\chi-\tau)^{\kappa-\rho_j} y_j(\tau)\, d\tau - \lambda_j \sum_{i=0}^{n'-1} \frac{\Gamma(\kappa+1) b_{j,i}}{\Gamma(\kappa - \rho_j + i + 2)} \chi^{1+i-\rho_j}. \tag{37}$$

Next, we multiplay by:

$$\tau^{\gamma_j/p_j} \varphi^{1/p_j} \cdot \tau^{-\gamma_j/p_j} \varphi^{-1/p_j}, j = 1,2$$

inside the integrals of (36) and (37) respectively , to get :

$$\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} y_j(\tau)\,d\tau = \int_0^\chi y_j(\tau)\varphi^{1/p_j}\tau^{\gamma_j/p_j}(\chi-\tau)^{\kappa-\alpha_j}\varphi^{-1/p_j}\tau^{-\gamma_j/p_j}\,d\tau, \quad (38)$$

$$\int_0^\chi (\chi-\tau)^{\kappa-\rho_j} y_j(\tau)\,d\tau = \int_0^\chi y_j(\tau)\varphi^{1/p_j}\tau^{\gamma_j/p_j}(\chi-\tau)^{\kappa-\rho_j}\varphi^{-1/p_j}\tau^{-\gamma_j/p_j}\,d\tau. \quad (39)$$

Applying Hölder's inequality, we get:

$$\begin{aligned}\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} y_j(\tau)\,d\tau &\leq \left(\int_0^\chi \varphi(\tau)\tau^{\gamma_j}|y_j(\tau)|^{p_j}\,d\tau\right)^{1/p_j} \\ &\times \left(\int_0^\chi \varphi(\tau)^{-p_j'/p_j}\tau^{-\gamma_j p_j'/p_j}(\chi-\tau)^{p_j'(\kappa-\alpha_j)}\,d\tau\right)^{1/p_j'} \\ &\leq I_j^{1/p_j}\left(\int_0^\chi \varphi^{-p_j'/p_j}\tau^{-\gamma_j p_j'/p_j}(\chi-\tau)^{p_j'(\kappa-\alpha_j)}\,d\tau\right)^{1/p_j'}, j=1,2\end{aligned}$$

and ,

$$\begin{aligned}\int_0^\chi (\chi-\tau)^{\kappa-\rho_j} y_j(\tau)\,d\tau &\leq \left(\int_0^\chi \varphi(\tau)\tau^{\gamma_j}|y_j(\tau)|^{p_j}\,d\tau\right)^{1/p_j} \quad (3.3) \\ &\times \left(\int_0^\chi \varphi(\tau)^{-p_j'/p_j}\tau^{-\gamma_j p_j'/p_j}(\chi-\tau)^{p_j'(\kappa-\rho_j)}\,d\tau\right)^{1/p_j'} \\ &\leq I_j^{1/p_j}\left(\int_0^\chi \varphi^{-p_j'/p_j}\tau^{-\gamma_j p_j'/p_j}(\chi-\tau)^{p_j'(\kappa-\rho_j)}\,d\tau\right)^{1/p_j'}, \quad j=1,2. \quad (40)\end{aligned}$$

Now, by Lemma 49, we find:

$$\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} y_j(\tau)\,d\tau \leq \left(K^{\kappa,\alpha_j}_{\gamma_j,p_j'}\right)^{1/p_j'} I_j^{1/p_j}\chi^{\kappa+\frac{1}{p_j'}-\alpha_j+\frac{\gamma_j}{p_j'}-\gamma_j}, \quad j=1,2,$$

By Lemma 49 , we have the estimate

$$\int_0^\chi (\chi-\tau)^{\kappa-\rho_j} y_j(\tau)\,d\tau \leq \left[K^{\gamma_j,p_j'}_{\kappa,\rho_j}\right]^{1/p_j'} I_j^{1/p_j}\chi^{\kappa+\frac{1}{p_j'}-\rho_j+\frac{\gamma_j}{p_j'}-\gamma_j}, \quad j=1,2. \quad (3.4)$$

and

$$\int_0^\chi (\chi-\tau)^{\kappa-\rho_j} y_j(\tau)\,d\tau \leq \left[K^{\gamma_j,p_j'}_{\kappa,\rho_j}\right]^{1/p_j'} I_j^{1/p_j}\chi^{\kappa+\frac{1}{p_j'}-\rho_j+\frac{\gamma_j}{p_j'}-\gamma_j}, \quad j=1,2.$$

then

$$J_j \leq K_j I_j^{1/p_j} \chi^{\frac{1}{p'_j}-\alpha_j+\frac{\gamma_j}{p'_j}-\gamma_j} - \sum_{i=0}^{n-1} \frac{\Gamma(\kappa+1) b_{j,i}}{\Gamma(\kappa-\alpha_j+i+2)} \chi^{1+i-\alpha_j}, \quad j=1,2, \tag{40}$$

$$H_j \leq K'_j I_j^{1/p_j} \chi^{\frac{1}{p'_j}-\rho_j+\frac{\gamma_j}{p'_j}-\gamma_j} - \lambda_j \sum_{i=0}^{n'-1} \frac{\Gamma(\kappa+1) b_{j,i}}{\Gamma(\kappa-\rho_j+i+2)} \chi^{1+i-\rho_j}, \quad j=1,2. \tag{41}$$

with

$$K_j = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\alpha_j)} \left[K_{\kappa,\alpha_j}^{\gamma_j,p'_j}\right]^{1/p'_j}, \quad j=1,2,$$
$$K'_j = |\lambda_j| \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\rho_j)} \left[K_{\kappa,\rho_j}^{\gamma_j,p'_j}\right]^{1/p'_j}, \quad j=1,2.$$

As $b_{j,i} \geq 0$, we find from (40) and (41)

$$J_j \leq K_j \chi^{\frac{1}{p'_j}-\alpha_j+\frac{\gamma_j}{p'_j}-\gamma_j} I_j^{1/p_j}, \quad j=1,2, \tag{42}$$

$$H_j \leq K'_j \chi^{\frac{1}{p'_j}-\rho_j+\frac{\gamma_j}{p'_j}-\gamma_j} I_j^{1/p_j}, \quad j=1,2. \tag{43}$$

From (29), (30), (42), and (43), we have:

$$I_2 \leq K''_1 \chi^{\frac{1}{p'_1}+\frac{\gamma_1}{p'_1}-\gamma_1} I_1^{1/p_1} \left(\chi^{-\alpha_1}+\chi^{-\rho_1}\right), \tag{44}$$

$$I_1 \leq K''_2 \chi^{\frac{1}{p'_2}+\frac{\gamma_2}{p'_2}-\gamma_2} I_2^{1/p_2} \left(\chi^{-\alpha_2}+\chi^{-\rho_2}\right), \tag{45}$$

where

$$K''_j = \max\{K_j, K'_j\}, \quad j=1,2.$$

consequently(44) and (45) become :

$$I_2^{1-\frac{1}{p_1p_2}} \leq K''_1 (K''_2)^{1/p_1} \chi^{\frac{1}{p'_1}+\frac{\gamma_1}{p'_1}-\gamma_1+\frac{1}{p_1p'_2}+\frac{\gamma_2}{p_1p'_2}-\frac{\gamma_2}{p_1}} \left(\chi^{-\alpha_2/p_1}+\chi^{-\rho_2/p_1}\right)\left(\chi^{-\alpha_1}+\chi^{-\rho_1}\right), \tag{3.5}$$

$$I_1^{1-\frac{1}{p_1p_2}} \leq K''_2 (K''_1)^{1/p_2} \chi^{\frac{1}{p'_2}+\frac{\gamma_2}{p'_2}-\gamma_2+\frac{1}{p_2p'_1}+\frac{\gamma_1}{p_2p'_1}-\frac{\gamma_1}{p_2}} \left(\chi^{-\alpha_1/p_2}+\chi^{-\rho_1/p_2}\right)\left(\chi^{-\alpha_2}+\chi^{-\rho_2}\right). \tag{3.6}$$

or

$$I_2^{1-\frac{1}{p_1p_2}} \leq K''_1 \left(K''_2\right)^{\frac{1}{p_1}} \left(\chi^{s_1}+\chi^{s_2}+\chi^{s_3}+\chi^{s_4}\right), \tag{46}$$

and

$$I_1^{1-\frac{1}{p_1p_2}} \leq K_2''\left(K_1''\right)^{\frac{1}{p_2}}\left(\chi^{s_5}+\chi^{s_6}+\chi^{s_7}+\chi^{s_8}\right). \tag{47}$$

where We have used Lemma 39 with $0 \leq r \leq 1$ and:

$$\begin{aligned}
s_1 &= -\frac{\alpha_2}{p_1}-\alpha_1+\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1+\frac{1}{p_1p_2'}+\frac{\gamma_2}{p_1p_2'}-\frac{\gamma_2}{p_1},\\
s_2 &= -\frac{\alpha_2}{p_1}-\rho_1+\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1+\frac{1}{p_1p_2'}+\frac{\gamma_2}{p_1p_2'}-\frac{\gamma_2}{p_1},\\
s_3 &= -\frac{\rho_2}{p_1}-\alpha_1+\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1+\frac{1}{p_1p_2'}+\frac{\gamma_2}{p_1p_2'}-\frac{\gamma_2}{p_1},\\
s_4 &= -\frac{\rho_2}{p_1}-\rho_1+\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1+\frac{1}{p_1p_2'}+\frac{\gamma_2}{p_1p_2'}-\frac{\gamma_2}{p_1},\\
s_5 &= -\frac{\alpha_1}{p_2}-\alpha_2+\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2+\frac{1}{p_2p_1'}+\frac{\gamma_1}{p_2p_1'}-\frac{\gamma_1}{p_2},\\
s_6 &= -\frac{\alpha_1}{p_2}-\rho_2+\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2+\frac{1}{p_2p_1'}+\frac{\gamma_1}{p_2p_1'}-\frac{\gamma_1}{p_2},\\
s_7 &= -\frac{\rho_1}{p_2}-\alpha_2+\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2+\frac{1}{p_2p_1'}+\frac{\gamma_1}{p_2p_1'}-\frac{\gamma_1}{p_2},\\
s_8 &= -\frac{\rho_1}{p_2}-\rho_2+\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2+\frac{1}{p_2p_1'}+\frac{\gamma_1}{p_2p_1'}-\frac{\gamma_1}{p_2}.
\end{aligned}$$

If

$$1-\frac{1}{p_1p_2} < \frac{\rho_2}{p_1}+\rho_1+\frac{\gamma_1}{p_1}+\frac{\gamma_2}{p_1p_2},$$

then $s_m < 0$ and consequently $\chi^{s_m} \to 0$ as $\chi \to \infty$, for $m = 1,2,3,4$.
Similarly, $s_m < 0$, $m = 5,6,7,8$, if

$$1-\frac{1}{p_1p_2} < \frac{\rho_1}{p_2}+\rho_2+\frac{\gamma_2}{p_2}+\frac{\gamma_1}{p_1p_2},$$

and therefore $\chi^{s_m} \to 0$, $m = 5,6,7,8$, as $\chi \to \infty$. Thus,

$$\lim_{\chi\to\infty}\int_0^{\chi}\tau^{\gamma_j}\phi(\tau)|u_j(\tau)|^{p_j}\,d\tau = 0, \quad j = 1,2.$$

This means that $u_j = 0$, $j = 1,2$,we come to a contradiction contradicts the assumption.

Theorem 54. *Assume that*

$$1-\frac{1}{p_1p_2}\leq\frac{\rho_1+\rho_2}{p_1}+\frac{\gamma_1}{p_1}+\frac{\gamma_2}{p_1p_2}\quad or\quad 1-\frac{1}{p_1p_2}\leq\frac{\rho_2+\rho_1}{p_2}+\frac{\gamma_2}{p_2}+\frac{\gamma_1}{p_1p_2},$$

where

$$(\alpha_j-n-\rho_j+1)p_j-1<\gamma_j<p_j-1,\quad j=1,2.$$

Then, the system (27) admits no global nontrivial solutions in the space $C_{\alpha_1}[0,\infty)\times C_{\alpha_2}[0,\infty)$ *when* $b_{j,n-1}>0$, $j=1,2$.

Proof. From the previous theorem (see (46) and (47)) we obtained:

$$I_2^{1-\frac{1}{p_1p_2}}\leq K_1''(K_2'')^{1/p_1}(\chi^{s_1}+\chi^{s_2}+\chi^{s_3}+\chi^{s_4}),$$

and

$$I_1^{1-\frac{1}{p_1p_2}}\leq K_2''(K_1'')^{1/p_2}(\chi^{s_5}+\chi^{s_6}+\chi^{s_7}+\chi^{s_8}).$$

□

In this case, if

$$1-\frac{1}{p_1p_2}\leq\frac{\rho_1+\rho_2}{p_1}+\frac{\gamma_1}{p_1}+\frac{\gamma_2}{p_1p_2},$$

then $s_m\leq 0$ for $m=1,2,3,4$; or if

$$1-\frac{1}{p_1p_2}\leq\frac{\rho_2+\rho_1}{p_2}+\frac{\gamma_2}{p_2}+\frac{\gamma_1}{p_1p_2},$$

then $s_m\leq 0$ for $m=5,6,7,8$; and therefore I_1 and I_2 are bounded.

Now, from equations (11), (12), (17), and (20), we have:

$$I_2\leq K_1\chi^{\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1}I_1^{\frac{1}{p_1}}+K_1'\chi^{\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1-\rho_1+\alpha_1}I_1^{\frac{1}{p_1}}-\sum_{i=0}^{n-1}\frac{\Gamma(\kappa+1)\,b_{1,i}}{\Gamma(\kappa+1-i)}\chi^{-i},$$

and similarly,

$$I_1\leq K_2\chi^{\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2}I_2^{\frac{1}{p_2}}+K_2'\chi^{\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2-\rho_2+\alpha_2}I_2^{\frac{1}{p_2}}-\sum_{i=0}^{n-1}\frac{\Gamma(\kappa+1)\,b_{2,i}}{\Gamma(\kappa+1-i)}\chi^{-i}.$$

Or, equivalently,

$$I_2+b_{1,0}\leq\max\{K_1,K_1'\}\chi^{\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1}\left(\chi^{-\alpha_1}+\chi^{-\rho_1}\right)I_1^{\frac{1}{p_1}}-\sum_{i=1}^{n-1}\frac{\Gamma(\kappa+1)\,b_{1,i}}{\Gamma(\kappa+1-i)}\chi^{-i},$$

$$I_1+b_{2,0}\leq\max\{K_2,K_2'\}\chi^{\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2}\left(\chi^{-\alpha_2}+\chi^{-\rho_2}\right)I_2^{\frac{1}{p_2}}-\sum_{i=1}^{n-1}\frac{\Gamma(\kappa+1)\,b_{2,i}}{\Gamma(\kappa+1-i)}\chi^{-i}.$$

Since $I_j \geq 0$ for $j = 1,2$, then:

$$b_{1,0} \leq \max\{K_1, K_1'\}\chi^{\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1}\left(\chi^{-\alpha_1}+\chi^{-\rho_1}\right)I_1^{\frac{1}{p_1}} - \sum_{i=1}^{n-1}\frac{\Gamma(\kappa+1)\,b_{1,i}}{\Gamma(\kappa+1-i)}\chi^{-i},$$

$$b_{2,0} \leq \max\{K_2, K_2'\}\chi^{\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2}\left(\chi^{-\alpha_2}+\chi^{-\rho_2}\right)I_2^{\frac{1}{p_2}} - \sum_{i=1}^{n-1}\frac{\Gamma(\kappa+1)\,b_{2,i}}{\Gamma(\kappa+1-i)}\chi^{-i}.$$

When $\chi \to \infty$, we obtain the contradiction:

$$0 < b_{j,0} \leq 0, \quad j = 1,2.$$

□

4 Non-existence of solutions in the case of CFD

In this part, we discuss the system:

$$\begin{cases} {}^C D_0^{\alpha_1} y_1(\tau) + \lambda_1 \, {}^C D_0^{\rho_1} y_1(\tau) \geq \tau^{\gamma_2} |y_2(\tau)|^{p_2}, & 0 < \rho_1 < \alpha_1,\ \tau > 0, \\ {}^C D_0^{\alpha_2} y_2(\tau) + \lambda_2 \, {}^C D_0^{\rho_2} y_2(\tau) \geq \tau^{\gamma_1} |y_1(\tau)|^{p_1}, & 0 < \rho_2 < \alpha_2,\ \tau > 0, \end{cases} \tag{27}$$

with initial conditions:

$$y_j^{(k)}(0) = b_{j,k}, \quad k = 0,\dots,n_j-1, \quad n_j = \lceil \alpha_j \rceil, \quad b_{j,k} \in \mathbb{R}, \quad j = 1,2. \tag{28}$$

where ${}^C D_0^{\sigma}$ denotes the Caputo fractional derivative, $p_j > 1$ and $\gamma_j \in \mathbb{R}$, for $j = 1,2$.

Theorem 55 (Theorem)**.** *Assume that*

$$1 - \frac{1}{p_1 p_2} < \frac{\rho_2}{p_1} + \rho_1 + \frac{\gamma_1}{p_1} + \frac{\gamma_2}{p_1 p_2} \quad \text{or} \quad 1 - \frac{1}{p_1 p_2} < \frac{\rho_1}{p_2} + \rho_2 + \frac{\gamma_2}{p_2} + \frac{\gamma_1}{p_1 p_2},$$

where $p_j > 1$ and $\gamma_j < p_j - 1$, for $j = 1,2$. Then, Problem (27)–(28) does not admit nontrivial global solutions in the space

$$C^{\alpha_1}[0,\infty) \times C^{\alpha_2}[0,\infty),$$

where $C^{\alpha}[0,\infty)$ is defined as in (4), provided that $b_{j,k} \geq 0$, for $j = 1,2$.

Proof. We argue by contradiction. Suppose that a global solution (y_1, y_2) exists for all $\tau > 0$. Let φ be as in (7) with

$$\kappa > \max\left\{0,\, \alpha_j - 1,\, \frac{p_j \alpha_j}{p_j - 1} - 1\right\}, \quad j = 1,2.$$

Multiplying (27) by φ and integrating over the interval $[0,\chi]$, we obtain:

$$\int_0^\chi \varphi(\tau)\,\tau^{\gamma_2}|y_2(\tau)|^{p_2}\,d\tau \le \int_0^\chi \varphi(\tau)\,{}^C D_0^{\alpha_1} y_1(\tau)\,d\tau + \lambda_1 \int_0^\chi \varphi(\tau)\,{}^C D_0^{\rho_1} y_1(\tau)\,d\tau, \tag{29}$$

$$\int_0^\chi \varphi(\tau)\,\tau^{\gamma_1}|y_1(\tau)|^{p_1}\,d\tau \le \int_0^\chi \varphi(\tau)\,{}^C D_0^{\alpha_2} y_2(\tau)\,d\tau + \lambda_2 \int_0^\chi \varphi(\tau)\,{}^C D_0^{\rho_2} y_2(\tau)\,d\tau. \tag{30}$$

Define:

$$I_j = \int_0^\chi \varphi(\tau)\,\tau^{\gamma_j}|y_j(\tau)|^{p_j}\,d\tau, \quad j=1,2, \tag{31}$$

$$J_j = \int_0^\chi \varphi(\tau)\,{}^C D_0^{\alpha_j} y_j(\tau)\,d\tau, \quad j=1,2, \tag{32}$$

$$H_j = \lambda_j \int_0^\chi \varphi(\tau)\,{}^C D_0^{\rho_j} y_j(\tau)\,d\tau, \quad j=1,2. \tag{33}$$

Let $n' = \lceil \rho_j \rceil$, for $j=1,2$. By Lemma 47 we get:

$$J_j = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\alpha_j)}\chi^{-\kappa}\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} y_j(\tau)\,d\tau \tag{3.7}$$

$$-\sum_{i=0}^{n-1} \frac{\Gamma(\kappa+1)\,b_{j,i}}{\Gamma(\kappa-\alpha_j+i+2)}\chi^{1+i-\alpha_j}, \quad j=1,2. \tag{36}$$

$$H_j = \lambda_j \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\rho_j)}\chi^{-\kappa}\int_0^\chi (\chi-\tau)^{\kappa-\rho_j} y_j(\tau)\,d\tau \tag{3.8}$$

$$-\lambda_j \sum_{i=0}^{n'-1} \frac{\Gamma(\kappa+1)\,b_{j,i}}{\Gamma(\kappa-\rho_j+i+2)}\chi^{1+i-\rho_j}, \quad j=1,2. \tag{37}$$

Next, we multiply inside the integrals in (36) and (37) by

$$\frac{\tau^{\gamma_j/p_j}\phi^{1/p_j}}{\tau^{\gamma_j/p_j}\phi^{1/p_j}} = 1, \quad j=1,2,$$

to obtain:

$$\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} y_j(\tau)\,d\tau = \int_0^\chi y_j(\tau)\,\phi^{1/p_j}\tau^{\gamma_j/p_j}(\chi-\tau)^{\kappa-\alpha_j}\phi^{-1/p_j}\tau^{-\gamma_j/p_j}\,d\tau, \tag{38}$$

$$\int_0^\chi (\chi-\tau)^{\kappa-\rho_j} y_j(\tau)\,d\tau = \int_0^\chi y_j(\tau)\,\phi^{1/p_j}\tau^{\gamma_j/p_j}(\chi-\tau)^{\kappa-\rho_j}\phi^{-1/p_j}\tau^{-\gamma_j/p_j}\,d\tau. \tag{39}$$

Applying Hölder's inequality, we obtain:

$$\begin{aligned}\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} y_j(\tau)\,d\tau &\le \left(\int_0^\chi \phi(\tau)\,\tau^{\gamma_j}|y_j(\tau)|^{p_j}d\tau\right)^{1/p_j} \\ &\quad\times\left(\int_0^\chi \phi^{-p_j'/p_j}\tau^{-\gamma_j p_j'/p_j}(\chi-\tau)^{p_j'(\kappa-\alpha_j)}d\tau\right)^{1/p_j'} \\ &= I_j^{1/p_j}\left(\int_0^\chi \phi^{-p_j'/p_j}\tau^{-\gamma_j p_j'/p_j}(\chi-\tau)^{p_j'(\kappa-\alpha_j)}d\tau\right)^{1/p_j'}, \quad j=1,2.\end{aligned}$$

and similarly:

$$\int_0^\chi (\chi-\tau)^{\kappa-\rho_j} y_j(\tau)\,d\tau \le I_j^{1/p_j}\left(\int_0^\chi \phi^{-p_j'/p_j}\tau^{-\gamma_j p_j'/p_j}(\chi-\tau)^{p_j'(\kappa-\rho_j)}d\tau\right)^{1/p_j'}, \quad j=1,2.$$

Now, by Lemma 49 , we obtain:

$$\int_0^\chi (\chi-\tau)^{\kappa-\alpha_j} y_j(\tau)\,d\tau \le \left[K_{\gamma_j,p_j'}^{\kappa,\alpha_j}\right]^{1/p_j'} I_j^{1/p_j}\chi^{\kappa+\frac{1}{p_j'}-\alpha_j+\frac{\gamma_j}{p_j'}-\gamma_j}, \quad j=1,2.$$

Continuing from the previous derivation, we also have:

$$\int_0^\chi (\chi-\tau)^{\kappa-\rho_j} y_j(\tau)\,d\tau \le \left[K_{\gamma_j,p_j'}^{\kappa,\rho_j}\right]^{1/p_j'} I_j^{1/p_j}\chi^{\kappa+\frac{1}{p_j'}-\rho_j+\frac{\gamma_j}{p_j'}-\gamma_j}, \quad j=1,2.$$

Then

$$J_j \le K_j I_j^{1/p_j}\chi^{\frac{1}{p_j'}-\alpha_j+\frac{\gamma_j}{p_j'}-\gamma_j} - \sum_{i=0}^{n-1}\frac{\Gamma(\kappa+1)b_{j,i}}{\Gamma(\kappa-\alpha_j+i+2)}\chi^{1+i-\alpha_j}, \quad j=1,2. \tag{40}$$

$$H_j \le K_j' I_j^{1/p_j}\chi^{\frac{1}{p_j'}-\rho_j+\frac{\gamma_j}{p_j'}-\gamma_j} - \lambda_j\sum_{i=0}^{n'-1}\frac{\Gamma(\kappa+1)b_{j,i}}{\Gamma(\kappa-\rho_j+i+2)}\chi^{1+i-\rho_j}, \quad j=1,2. \tag{41}$$

with

$$K_j = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\alpha_j)}\left[K_{\gamma_j,p_j'}^{\kappa,\alpha_j}\right]^{1/p_j'}, \quad j=1,2,$$

$$K'_j = |\lambda_j| \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\rho_j)} \left[K^{\kappa,\rho_j}_{\gamma_j,p'_j} \right]^{1/p'_j}, \quad j = 1,2.$$

Since $b_{j,i} \geq 0$, for $j = 1,2$, we obtain from (40) and (41):

$$J_j \leq K_j \chi^{\frac{1}{p'_j} - \alpha_j + \frac{\gamma_j}{p'_j} - \gamma_j} I_j^{1/p_j}, \quad j = 1,2, \tag{42}$$

$$H_j \leq K'_j \chi^{\frac{1}{p'_j} - \rho_j + \frac{\gamma_j}{p'_j} - \gamma_j} I_j^{1/p_j}, \quad j = 1,2. \tag{43}$$

From equations (29), (30), (42), and (43), we obtain:

$$I_2 \leq K''_1 \chi^{\frac{1}{p'_1} + \frac{\gamma_1}{p'_1} - \gamma_1} I_1^{1/p_1} (\chi^{-\alpha_1} + \chi^{-\rho_1}), \tag{44}$$

$$I_1 \leq K''_2 \chi^{\frac{1}{p'_2} + \frac{\gamma_2}{p'_2} - \gamma_2} I_2^{1/p_2} (\chi^{-\alpha_2} + \chi^{-\rho_2}), \tag{45}$$

where

$$K''_j = \max\{K_j, K'_j\}, \quad j = 1,2.$$

Consequently, (44) and (45) become: Consequently, equations (44) and (45) become:

$$I_2^{1-\frac{1}{p_1p_2}} \leq K''_1 \left(K''_2\right)^{1/p_1} \chi^{\frac{1}{p'_1} + \frac{\gamma_1}{p'_1} - \gamma_1 + \frac{1}{p_1p'_2} + \frac{\gamma_2}{p_1p'_2} - \frac{\gamma_2}{p_1}} \left(\chi^{-\alpha_2/p_1} + \chi^{-\rho_2/p_1}\right) \left(\chi^{-\alpha_1} + \chi^{-\rho_1}\right), \tag{3.9}$$

$$I_1^{1-\frac{1}{p_1p_2}} \leq K''_2 \left(K''_1\right)^{1/p_2} \chi^{\frac{1}{p'_2} + \frac{\gamma_2}{p'_2} - \gamma_2 + \frac{1}{p_2p'_1} + \frac{\gamma_1}{p_2p'_1} - \frac{\gamma_1}{p_2}} \left(\chi^{-\alpha_1/p_2} + \chi^{-\rho_1/p_2}\right) \left(\chi^{-\alpha_2} + \chi^{-\rho_2}\right), \tag{3.10}$$

or, more compactly:

$$I_2^{1-\frac{1}{p_1p_2}} \leq K''_1 \left(K''_2\right)^{1/p_1} \left(\chi^{s_1} + \chi^{s_2} + \chi^{s_3} + \chi^{s_4}\right), \tag{46}$$

$$I_1^{1-\frac{1}{p_1p_2}} \leq K''_2 \left(K''_1\right)^{1/p_2} \left(\chi^{s_5} + \chi^{s_6} + \chi^{s_7} + \chi^{s_8}\right), \tag{47}$$

Where we have used Lemma 39 with $0 \leq r \leq 1$, and

$$\begin{aligned}
s_1 &= -\frac{\alpha_2}{p_1} - \alpha_1 + \frac{1}{p_1'} + \frac{\gamma_1}{p_1'} - \gamma_1 + \frac{1}{p_1 p_2'} + \frac{\gamma_2}{p_1 p_2'} - \frac{\gamma_2}{p_1},\\
s_2 &= -\frac{\alpha_2}{p_1} - \rho_1 + \frac{1}{p_1'} + \frac{\gamma_1}{p_1'} - \gamma_1 + \frac{1}{p_1 p_2'} + \frac{\gamma_2}{p_1 p_2'} - \frac{\gamma_2}{p_1},\\
s_3 &= -\frac{\rho_2}{p_1} - \alpha_1 + \frac{1}{p_1'} + \frac{\gamma_1}{p_1'} - \gamma_1 + \frac{1}{p_1 p_2'} + \frac{\gamma_2}{p_1 p_2'} - \frac{\gamma_2}{p_1},\\
s_4 &= -\frac{\rho_2}{p_1} - \rho_1 + \frac{1}{p_1'} + \frac{\gamma_1}{p_1'} - \gamma_1 + \frac{1}{p_1 p_2'} + \frac{\gamma_2}{p_1 p_2'} - \frac{\gamma_2}{p_1},\\
s_5 &= -\frac{\alpha_1}{p_2} - \alpha_2 + \frac{1}{p_2'} + \frac{\gamma_2}{p_2'} - \gamma_2 + \frac{1}{p_2 p_1'} + \frac{\gamma_1}{p_2 p_1'} - \frac{\gamma_1}{p_2},\\
s_6 &= -\frac{\alpha_1}{p_2} - \rho_2 + \frac{1}{p_2'} + \frac{\gamma_2}{p_2'} - \gamma_2 + \frac{1}{p_2 p_1'} + \frac{\gamma_1}{p_2 p_1'} - \frac{\gamma_1}{p_2},\\
s_7 &= -\frac{\rho_1}{p_2} - \alpha_2 + \frac{1}{p_2'} + \frac{\gamma_2}{p_2'} - \gamma_2 + \frac{1}{p_2 p_1'} + \frac{\gamma_1}{p_2 p_1'} - \frac{\gamma_1}{p_2},\\
s_8 &= -\frac{\rho_1}{p_2} - \rho_2 + \frac{1}{p_2'} + \frac{\gamma_2}{p_2'} - \gamma_2 + \frac{1}{p_2 p_1'} + \frac{\gamma_1}{p_2 p_1'} - \frac{\gamma_1}{p_2}.
\end{aligned}$$

if:

$$1 - \frac{1}{p_1 p_2} < \frac{\rho_2}{p_1} + \rho_1 + \frac{\gamma_1}{p_1} + \frac{\gamma_2}{p_1 p_2}.$$

Then all $s_m < 0$ for $m = 1,2,3,4$, and hence

$$\chi^{s_m} \to 0 \quad \text{as } \chi \to \infty.$$

Similarly, if

$$1 - \frac{1}{p_1 p_2} < \frac{\rho_1}{p_2} + \rho_2 + \frac{\gamma_2}{p_2} + \frac{\gamma_1}{p_1 p_2},$$

then $s_m < 0$ for $m = 5,6,7,8$, and again

$$\chi^{s_m} \to 0 \quad \text{as } \chi \to \infty.$$

thus

$$\lim_{\chi\to\infty} \int_0^{\chi} \tau^{\gamma_j} \varphi(\tau) |u_j(\tau)|^{p_j} d\tau = 0, \quad j = 1,2.$$

This means $u_j = 0$, $j = 1,2$, we come to a contradiction .

Theorem 56. *Assume that*

$$1 - \frac{1}{p_1 p_2} \le \frac{\rho_1 + \rho_2}{p_1} + \frac{\gamma_1}{p_1} + \frac{\gamma_2}{p_1 p_2} \quad \text{or} \quad 1 - \frac{1}{p_1 p_2} \le \frac{\rho_2 + \rho_1}{p_2} + \frac{\gamma_2}{p_2} + \frac{\gamma_1}{p_1 p_2},$$

where $(\alpha_j - n - \rho_j + 1)p_j - 1 < \gamma_j < p_j - 1$, $j = 1,2$. *Then the system (27) admits no global nontrivial solutions in the space* $C^{\alpha_1}[0,\infty) \times C^{\alpha_2}[0,\infty)$ *when* $b_{j,n-1} > 0$, $j = 1,2$.

Proof. From the preceding analysis, and using inequalities (46) and (47), we found:

$$I_2^{1-\frac{1}{p_1p_2}} \leq K_1''(K_2'')^{1/p_1}\left(\chi^{s_1}+\chi^{s_2}+\chi^{s_3}+\chi^{s_4}\right),$$
$$I_1^{1-\frac{1}{p_1p_2}} \leq K_2''(K_1'')^{1/p_2}\left(\chi^{s_5}+\chi^{s_6}+\chi^{s_7}+\chi^{s_8}\right).$$

in this case

If

$$1-\frac{1}{p_1p_2} \leq \frac{\rho_1+\rho_2}{p_1}+\frac{\gamma_1}{p_1}+\frac{\gamma_2}{p_1p_2},$$

then $s_m \leq 0$ for $m = 1,2,3,4$, and similarly, if

$$1-\frac{1}{p_1p_2} \leq \frac{\rho_2+\rho_1}{p_2}+\frac{\gamma_2}{p_2}+\frac{\gamma_1}{p_1p_2},$$

then $s_m \leq 0$ for $m = 5,6,7,8$. In both cases, the integrals I_1 and I_2 are bounded.

From inequalities (29), (30), (40), and (41), we obtain:

$$\begin{aligned} I_2 \leq{}& K_1\chi^{\frac{1}{p_1'}-\alpha_1+\frac{\gamma_1}{p_1'}-\gamma_1} I_1^{1/p_1} + K_1'\chi^{\frac{1}{p_1'}-\rho_1+\frac{\gamma_1}{p_1'}-\gamma_1} I_1^{1/p_1} \\ &-\sum_{i=0}^{n'-1}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa-\rho_1+i+2)}\chi^{1+i-\rho_1} - \sum_{i=0}^{n-1}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa-\alpha_1+i+2)}\chi^{-\alpha_1+i+1},\end{aligned}$$

and

$$\begin{aligned} I_1 \leq{}& K_2\chi^{\frac{1}{p_2'}-\alpha_2+\frac{\gamma_2}{p_2'}-\gamma_2} I_2^{1/p_2} + K_2'\chi^{\frac{1}{p_2'}-\rho_2+\frac{\gamma_2}{p_2'}-\gamma_2} I_2^{1/p_2} \\ &-\sum_{i=0}^{n'-1}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa-\rho_2+i+2)}\chi^{1+i-\rho_2} - \sum_{i=0}^{n-1}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa-\alpha_2+i+2)}\chi^{-\alpha_2+i+1}.\end{aligned}$$

or, we have:

$$\begin{aligned} &I_2+\frac{\Gamma(\kappa+1)b_{1,n'-1}}{\Gamma(\kappa-\rho_1+n'+1)}\chi^{n'-\rho_1}+\frac{\Gamma(\kappa+1)b_{1,n-1}}{\Gamma(\kappa-\alpha_1+n+1)}\chi^{n-\alpha_1} \\ &\leq \max\{K_1,K_1'\}\chi^{\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1}\left(\chi^{-\alpha_1}+\chi^{-\rho_1}\right)I_1^{1/p_1} \\ &\quad -\sum_{i=0}^{n'-2}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa-\rho_1+i+2)}\chi^{1+i-\rho_1} - \sum_{i=0}^{n-2}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa-\alpha_1+i+2)}\chi^{-\alpha_1+i+1},\end{aligned}$$

and ,

$$\begin{aligned} &I_1+\frac{\Gamma(\kappa+1)b_{2,n'-1}}{\Gamma(\kappa-\rho_2+n'+1)}\chi^{n'-\rho_2}+\frac{\Gamma(\kappa+1)b_{2,n-1}}{\Gamma(\kappa-\alpha_2+n+1)}\chi^{n-\alpha_2} \\ &\leq \max\{K_2,K_2'\}\chi^{\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2}\left(\chi^{-\alpha_2}+\chi^{-\rho_2}\right)I_2^{1/p_2} \\ &\quad -\sum_{i=0}^{n'-2}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa-\rho_2+i+2)}\chi^{1+i-\rho_2} - \sum_{i=0}^{n-2}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa-\alpha_2+i+2)}\chi^{-\alpha_2+i+1}.\end{aligned}$$

Since $I_j \geq 0,\ j = 1,2$, we obtain:

$$\chi^{n-\alpha_1}\left[\frac{\Gamma(\kappa+1)b_{1,n-1}}{\Gamma(\kappa-\alpha_1+n+1)}+\frac{\Gamma(\kappa+1)b_{1,n'-1}}{\Gamma(\kappa-\rho_1+n'+1)}\chi^{n'-\rho_1-n+\alpha_1}\right] \leq \max\{K_1,K_1'\}\chi^{\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1} \times\left(\chi^{-\alpha_1}+\chi^{-\rho_1}\right)I_1^{1/p_1} - \sum_{i=0}^{n'-2}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa-\rho_1+i+2)}\chi^{1+i-\rho_1} - \sum_{i=0}^{n-2}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa-\alpha_1+i+2)}\chi^{-\alpha_1+i+1},$$

and,

$$\chi^{n-\alpha_2}\left[\frac{\Gamma(\kappa+1)b_{2,n-1}}{\Gamma(\kappa-\alpha_2+n+1)}+\frac{\Gamma(\kappa+1)b_{2,n'-1}}{\Gamma(\kappa-\rho_2+n'+1)}\chi^{n'-\rho_2-n+\alpha_2}\right] \leq \max\{K_2,K_2'\}\chi^{\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2} \times\left(\chi^{-\alpha_2}+\chi^{-\rho_2}\right)I_2^{1/p_2} - \sum_{i=0}^{n'-2}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa-\rho_2+i+2)}\chi^{1+i-\rho_2} - \sum_{i=0}^{n-2}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa-\alpha_2+i+2)}\chi^{-\alpha_2+i+1}.$$

We finally observe that:

$$\frac{\Gamma(\kappa+1)b_{1,n-1}}{\Gamma(\kappa-\alpha_1+n+1)} \leq \max\{K_1,K_1'\}\chi^{\alpha_1-n+\frac{1}{p_1'}+\frac{\gamma_1}{p_1'}-\gamma_1}\left(\chi^{-\alpha_1}+\chi^{-\rho_1}\right)I_1^{1/p_1}$$

$$-\chi^{\alpha_1-n}\sum_{i=0}^{n'-2}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa-\rho_1+i+2)}\chi^{1+i-\rho_1} - \chi^{\alpha_1-n}\sum_{i=0}^{n-2}\frac{\Gamma(\kappa+1)b_{1,i}}{\Gamma(\kappa-\alpha_1+i+2)}\chi^{-\alpha_1+i+1},$$

and similarly,

$$\frac{\Gamma(\kappa+1)b_{2,n-1}}{\Gamma(\kappa-\alpha_2+n+1)} \leq \max\{K_2,K_2'\}\chi^{\alpha_2-n+\frac{1}{p_2'}+\frac{\gamma_2}{p_2'}-\gamma_2}\left(\chi^{-\alpha_2}+\chi^{-\rho_2}\right)I_2^{1/p_2}$$

$$-\chi^{\alpha_2-n}\sum_{i=0}^{n'-2}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa-\rho_2+i+2)}\chi^{1+i-\rho_2} - \chi^{\alpha_2-n}\sum_{i=0}^{n-2}\frac{\Gamma(\kappa+1)b_{2,i}}{\Gamma(\kappa-\alpha_2+i+2)}\chi^{-\alpha_2+i+1}.$$

when $\chi \to \infty$, we obtain the contradiction:

$$0 < b_{j,n-1} \leq 0, \quad j = 1,2.$$

□

3.5 Conclusion general

Studying differential inequalities involving fractional derivatives reveals just how deep and complex the phenomena described by mathematics can be. Unlike classical derivatives, fractional derivatives take into account the influence of the past, giving us tools to understand processes that depend not only on the present moment but also on the full history of a system.

This new perspective opens up wide horizons for real-world applications in physics, biology, economics, and other fields. By analyzing these types of inequalities, we can build more accurate and stable models that allow for a better understanding of complex system behavior.

Although there is still a long way to go for researchers, the results achieved so far are promising and encourage further exploration. They remind us how abstract mathematical concepts can shed light on hidden aspects of the real world.

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